Informative regression model under random censorship from both sides and estimation of survival function

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In survival data analysis, response random variable (r.v.) Z , the survival time of a patient, that usually can be influenced by r.v. X, often called prognostic factor. In fact, in practical situations often occurs that not all of survival times $Z_1, ..., Z_n$ corresponding to n individuals, are completely observed, they may be censored. In this article we consider the case, when lifetimes censored from both sides. So let $\left\{(Z_k, L_k, Y_k, X_k)\,,\ k = \overline{1.n}\right\}$ are independent replicas of vector (Z, L, Y, X), where components of vector (Z, L, Y) are independent for given covariate X. Our sample will be consist of n vectors $S^{(n)} = \left\{\left(\xi_i, \chi_i^{(0)}, \chi_i^{(1)}, \chi_i^{(2)}, X_i\right)\,,\ 1 \leqslant i \leqslant n\right\}$ where $\xi_i = L_i \vee (Z_i \wedge Y_i)\,,\quad \chi_i^{(0)} = I\left(Z_i \wedge Y_i < L_i\right),\quad \chi_i^{(1)} = I\left(L_i \leqslant Z_i \leqslant Y_i\right),\ \chi_i^{(2)} = I\left(L_i \leqslant Y_i < Z_i\right)$ with $I\left(A\right)$ - denoting the indicator of event A, $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$. In sample $S^{(n)}$ the r.v.-s of interest Z_i are observable when $\chi_i^{(1)} = 1$. We denote by F_x, G_x and K_x the conditional distribution functions (d.f-s) of r.v.-s Z_x, Y_x and L_x respectively, given that $X = x$ and suppose that they are continuous. Let $0 \leqslant x_1 \leqslant x_2 \leqslant ... \leqslant x_n \leqslant 1$ denote n fixed design points of covariate X.

Let's H_x and N_x are conditional d.f-s of r.v.-s ξ_x and $\eta_x = Z_x \wedge Y_x$ for $X = x$. Then $H_x\left(t\right) = K_x\left(t\right) N_x\left(t\right)$ and $N_x\left(t\right) = 1 - \left(1 - F_x\left(t\right)\right)\left(1 - G_x\left(t\right)\right),\ t \geqslant 0$. We suppose that the censoring is informative so that the d.f.-s K_x and G_x are expressed from d.f. F_x with following formulas for all $t \geqslant 0$

$$\left\{ \begin{array}{c} 1 - G_x\left(t\right) = \left(1 - F_x\left(t\right)\right)^{\theta_x}, \\ K_x\left(t\right) = \left(N_x\left(t\right)\right)^{\beta_x}, \end{array} \right. \tag{1}$$

where θ_x and β_x are positive unknown parameters. Model (1) generalize well-known proportional hazards model of Koziol-Green [2], which follows from (1) for $\beta_x = 0$,in the absence of random censorship on the left $\left(K_x\left(t\right) \equiv 1\right)$ and without presence of covariate X. The following theorem characterizes the model (1).

Theorem 1. *Equalities (1) hold if and only if, when r.v. ξ_x and vector $\left(\chi_x^{(0)}, \chi_x^{(1)}, \chi_x^{(2)}\right)$ are independent.*

Should also be noted that model (1) in the absence of covariate X was investigated in [1]. Further assume that the model (1) holds. Then for survival function $1 - F_x$ has the representation $1 - F_x\left(t\right) = \left[1 - \left[H_x\left(x\right)\right]^{\lambda_x}\right]^{\gamma_x}, t \geqslant 0$, where $\lambda_x = \frac{1}{1+\beta_x}$ and $\gamma_x = \frac{1}{1+\theta_x}$. Using this representation, we construct the following estimate for $1 - F_x$:

$$1 - F_{xh}\left(t\right) = \left\{1 - \left[H_{xh}\left(t\right)\right]^{\lambda_{xh}}\right\}^{\gamma_{xh}},\ t \geqslant 0, \tag{2}$$

where $\lambda_{xh} = 1 - p_{xh}^{(0)}$, $\gamma_{xh} = p_{xh}^{(1)} \left(1 - p_{xh}^{(0)}\right)^{-1}$, $p_{xh}^{(m)} = \sum\limits_{i=1}^{n} \chi_i^{(m)} \omega_{ni}(x; h_n)$, $m = 0, 1, 2$, $H_{xh}(t) = \sum\limits_{i=1}^{n} I(\xi_i \leqslant t) \omega_{ni}(x; h_n)$,and $\{\omega_{ni}(x; h_n)\}_{i=1}^{n}$ are Gasser-Mullers weights.

$$\omega_{ni}(x; h_n) = \left(\int\limits_0^{x_n} \frac{1}{h_n} \pi \left(\frac{x-y}{h_n} \right) dy \right)^{-1} \int\limits_{x_{i-1}}^{x_i} \frac{1}{h_n} \pi \left(\frac{x-y}{h_n} \right) dy, \ i = 1, ..., n;$$

$x_o = 0$, $\pi(t)$ is known density function(kernell) and $h_n \downarrow 0, n \to \infty$. For the estimates (2) we have prove the following results:

(A) Exponential estimator for probability $P\left(\sup\limits_{\tau \leqslant x \leqslant T} |F_{xh}(t) - F_x(t)| > \varepsilon \right)$, where $\varepsilon > 0$, $\tau \leqslant T$;

(B) Strong uniform consistency with rate of convergence:

$$\sup_{\tau \leqslant x \leqslant T} |F_{xh}(t) - F_x(t)| \overset{\text{a.s.}}{=} O\left(\left(\frac{\log n}{nh_n} \right)^{1/2} \right);$$

(C) Asymptotic representation by sum of independent r.v.-s:

$$F_{xh}(t) - F_x(t) = \sum_{i=1}^{n} \omega_{ni}(x; h_n) \Psi_{tx}\left(\xi_i, \chi_i^{(0)}, \chi_i^{(1)}, \chi_i^{(2)}\right) + O\left(\frac{\log n}{nh_n} \right);$$

(D) Asymptotic normality:

$$(nh_n)^{1/2} (F_{xh}(t) - F_x(t)) \Rightarrow N\left(a_x(t), \sigma_x^2(t)\right).$$

(E) Weak convergence of stochastic process $\mathbb{W}_{nx}(t) = (nh_n)^{1/2}(F_{xh}(t) - F_x(t))$, $\tau \leqslant t \leqslant T$, in $\mathbb{D}[\tau, T]$ Gaussian process $\mathbb{W}_x(t)$ in Skorochod space

$$\mathbb{W}_{nx}(t) \overset{D}{\Rightarrow} \mathbb{W}_x(t) \text{ in } \mathbb{D}[\tau, T].$$

References

1. A. A. Abdushukurov. Model of random censorship from the both sides and independent tests for it. *D*oclady Acad. Nauk. Uzb. - 1994. - N.11. P. 8-9.(in Russian)
2. S. Csörgő. Estimating in proportional hazards model of random censorship. *S*tatistics. 1988. v.19. N.3. p. 437-463.

Mean residual life function estimation in the dependent models of random censorship

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In survival analysis (in engineering, in medical - biological researches) our interest focused on estimation of mean residual life function $e_n(x) = e_n(x,F) = E(X - x/X > x)$ of random variable (r.v.) X with survival function $S^X(x) = 1 - F(x) = P(X > x),\ x \geqslant 0,\ F(0) = 0$. However, in practical situation r.v. X is often censored. Consider the model of random censoring from the right, in which r.v. X with distribution function (d.f.) F is censored from the right by possible depending from X r.v. Y with d.f. $G, G(0) = 0$. Let $\{(X_i, Y_i)\,,\, i \geqslant 1\}$ is sequence of independent realization of pair (X, Y) with bivariate d.f. $H(x,y) = P(X \leqslant x, Y \leqslant y)$, $(x,y) \in \overline{R}^{+2} = [0,\infty]^2$. Observation is available the sample $V^{(n)} = \{(Z_i, \delta_i),\ 1 \leqslant i \leqslant n\}$, where $Z_i = \min(X_i, Y_i)$ and $\delta_i = I(Z_i = X_i)$, $I(A)$ is an indicator of the event A. In the sample $V^{(n)} = \{(Z_i, \delta_i),\ 1 \leqslant i \leqslant n\}$ the r.v.-s of interest X_i are observable only when $\delta_i = 1$.

Let's consider the following estimation for $e(x,F)$:

$$e_n(x) = e(x, F_n) = (S_n^X(x))^{-1} \cdot \int\limits_x^{+\infty} S_n^X(t)dt,\ x \in [0, Z^{(n)}].$$

Here $Z^{(n)} = \max(Z_1, ..., Z_n)$ and $S_n^X(x) = 1 - F_n(x)$ is copula-estimate for S^X from [1-3] is defined as

$$S_n^X(x) = \varphi^{-1}\left[\varphi(S_n^Z(x)) \cdot \frac{\int\limits_0^x I(S_n^Z(t-) > 0) S_n^Z(t-)\varphi'(S_n^Z(t)) d\Lambda_n^X(t)}{\int\limits_0^x I(S_n^Z(t-) > 0) S_n^Z(t-)\varphi'(S_n^Z(t)) d\Lambda_n^Z(t)}\right],$$

where

$$S_n^Z(x) = \frac{1}{n}\sum_{i=1}^n I(Z_i > x),$$

$$\Lambda_n^X(x) = \frac{1}{n}\sum_{i=1}^n \frac{I(Z_i \leqslant x,\, \delta_i = 1)}{S_n^Z(Z_i) + \frac{1}{n}},\ \ \Lambda_n^Z(x) = \frac{1}{n}\sum_{i=1}^n \frac{I(Z_i \leqslant x)}{S_n^Z(Z_i) + \frac{1}{n}},$$

φ is strong generator ($\varphi(0) = \infty$) of Archimedean copula survival function $C(u,v) = \varphi^{-1}[\varphi(u) + \varphi(v)]$, φ^{-1} is inverse of φ and by the Sclar's theorem for bivariate survival function $\overline{H}(x,y) = P(X > x, Y > y)$ we have representation

$$\varphi\left(\overline{H}(x,y)\right) = \varphi\left(S^X(x)\right) + \varphi\left(S^Y(y)\right),\ (x,y) \in \overline{R}^{+2}.$$

Let's introduce the weighted uniform measure $\varepsilon_n(F) = \sup\limits_{0\leqslant x<\infty} \chi(F(x)) \cdot |e(x;F_n) - e(x;F)|$, where weight function $\chi : [0,1] \to \overline{R}^+$ satisfies the following conditions:
(C1) Function χ is measurable and for every $\eta > 0 : \sup\{\chi(u) : u \in [0, 1-\eta]\} < \infty$;
(C2) Function $\chi^*(u) = \chi(u)/(1-u)$ is nondecreasing in a neighborhood of 1;
(C3) For $T_X = \sup\{x : S^X(x) > 0\}$, let

$$\int\limits_0^{T_X} \left\{ (S^X(x))^{-1} \int\limits_x^{T_X} \chi(F(y))dy \right\} dF(x) < \infty.$$

Let's enter also following regularity conditions for functions H, φ and Λ, where under Λ we mean the cumulative hazard functions $\Lambda^X = -\log S^X$ and $\Lambda^Z = -\log S^Z$:
(C4) The generator function $\varphi(\cdot)$ is strictly decreasing on (0,1] and is sufficiently smooth in the sense that the first two derivatives of the functions $\varphi(x)$ and $\psi(x) = -x\varphi^{'}(x)$ are bounded for $x \in [\varepsilon, 1]$, where $\varepsilon > 0$ is arbitrary. Moreover, the first derivative $\varphi^{'}$ is bounded away from zero on $[0,1]$;
(C5) $0 < \int\limits_0^{T_Z} [\psi(S^Z(x))]^2 d\Lambda(x) < \infty$;
(C6) $\int\limits_0^{T_Z} \left|\psi'(S^Z(x))\right| d\Lambda(x) < \infty$, $T_Z = \sup\{x : S^Z(x) > 0\}$.

Theorem. *Let $EX = e(0;F) < \infty$, conditions (C1)-(C6) are hold. Then for $n \to \infty$,*

$$\varepsilon_n(F) \xrightarrow{P} 0.$$

We discuss also the weak convergence of normed process $\sqrt{n}(e_n(x) - e(x))$ to the Gaussian process.

References

1. A. A. Abdushukurov. Survival function estimation in the dependent models of random censorship. *Book of abstracts of XXIX International Seminar on Stability Problems for Stochastic Models. Svetlogorsk*, Russia,10-16 October 2011. p.3-4.

2. A. A. Abdushukurov, N. T. Dushatov, R. S. Muradov, Estimating of functionals of a multidimensional distribution by censored observations with using copula functions. *In: Statistical Methods of Estimation and Hyphoteses testing.Perm State University.*, Perm. Issue 23, 2011, p. 36-47.(In Russian).

3. R. S. Muradov, A. A. Abdushukurov. Estimation of multivariate distributions and its mixtures by incomplete data. LAMBERT Academic Publishing, 2011, 123p.(In Russian).

Multi-channel queueing systems with various choice rules of channel for service

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We consider a multichannel queueing system S with r identical servers. The input $A(t)$ is a regenerative flow. It means that there exists an increasing sequence of random variables $\{\theta_j, j \geqslant 0\}$, $\theta_0 = 0$, such that $\{\theta_j - \theta_{j-1}, A(\theta_{j-1} + t) - A(\theta_{j-1}),\, t \in [0, \theta_j - \theta_{j-1})\}_{j=1}^{\infty}$ is a sequence of i.i.d. random elements. Then θ_j is the jth point of regeneration, $\tau_j = \theta_j - \theta_{j-1}$ is the jth regeneration interval and $\xi_j = A(\theta_j) - A(\theta_{j-1})$ is the number of customers arriving during τ_j. We suppose that $\mu = \mathsf{E}\tau_j < \infty$, $a = \mathsf{E}\xi_j < \infty, j \geqslant 1$. Then there exists $\lambda = \lim\limits_{t\to\infty} t^{-1}A(t)$ a.s. and λ is the intensity of the input flow $A(t)$. The service times $\{\eta_j\}_{j=1}^{\infty}$ are supposed to be i.i.d.r.v.'s with d.f. $B(x)$ and finite mean $b = \mathsf{E}\eta_j$. Besides, sequence $\{\eta_j\}_{j=1}^{\infty}$ does not depend on $A(t)$.

Note that the most part of flows used in queueing theory are regenerative. Doubly stochastic Poisson process with regenerative random intensity, Markov modulated, Semi-Markov flows and many others belong to this class. Besides, a regenerative flow has a collection useful properties.

We consider various choice rules of channel for service. For instance rule R_0 implies that there exists the general queue and customers are served by the first available server; in accordance with rule R_1 a server has its own queue and an arriving customer chooses a server with minimal queue in front of it; rule R_2 yields that an arriving customer is served by the jth server with probability $1/r$ independently of others; rule R_3 proposes that the nth customer is directed for service to the jth server if $n = rm + j$, where $m = 0, 1, \ldots,\ j = \overline{1, r}$.
Let $q_i(t)$ $(i = \overline{1, r})$ be the number of customers either waiting or being served on the ith channel at time t, $\overrightarrow{q(t)} = (q_1(t), \ldots, q_r(t))$ and $Q(t) = \sum\limits_{i=1}^{r} q_i(t)$. The corresponding workload processes are denoted by $W_i(t)$ and $\overrightarrow{W(t)} = (W_1(t), \ldots, W_r(t))$. Let t_n be the arrival time of the nth customer. We also consider imbedded processes $\overrightarrow{q_n} = \overrightarrow{q}(t_n - 0)$ and $\overrightarrow{W_n} = \overrightarrow{W}(t_n - 0)$.

For a choice rule R we introduce $\gamma_R(t) = \max_{i\leqslant r,\, j\leqslant r}(W_i(t) - W_j(t))$. Let $\mathbf{K_0}$ be the class of choice rules for which the stochastic process $\gamma_R(t)$ is stochastically bounded as $t \to \infty$ for every λ and b.
Assumption 1. $\mathsf{P}\,(\xi_1 < 1,\, \tau_1 < \eta_1) > 0$.
Assumption 2. The distribution of the regeneration interval τ_j of the input

flow $A(t)$ has an absolutely continuous component.

Theorem 1. If Assumption 1 holds then for any rule of choice from class $\mathbf{K_0}$ processes $\overrightarrow{W}_n$ and $\overrightarrow{Q}_n$ are ergodic if $\rho = \lambda b r^{-1} < 1$. If Assumptions 1 and 2 are fulfilled then it is true for $\overrightarrow{W}(t)$ and $\overrightarrow{Q}(t)$. If $\rho \geqslant 1$ then all these processes are stochastically unbounded.

Corollary 1. Theorem 1 is valid for queueing systems with choice rules $R_0 - R_3$.

Theorem 2. If $\mathsf{E}\tau_j^{2+\delta} < \infty$, $\mathsf{E}\xi_j^{2+\delta} < \infty$, $\mathsf{E}\eta_j^{2+\delta} < \infty$ and $\rho > 1$ then for any choice rule from class $\mathbf{K_0}$ the process

$$\widehat{Q}_T(t) = \frac{Q(tT) - (\rho - 1) r b^{-1} tT}{\sigma_Q \sqrt{tT}}$$

C-converges as $T \to \infty$ to Wiener's process on any finite interval $[\alpha, \beta]$. Here $\sigma_Q^2 = \sigma_A^2 + r\sigma_\eta^2 b^{-3}$, $\sigma_A^2 = \mu^{-1}\sigma_\xi^2 + \mu^{-3}a^2\sigma_\tau^2 - 2a\mu\mathsf{cov}(\tau_1, \xi_1)$.
If $\rho = 1$ then we get convergence to absolute value of Wiener's process.

Consider the case $\rho < 1$. We study the time-compression asymptotics. The input $A_\varepsilon(t)$ for S_i^ε is defined by the relation $A_\varepsilon(t) = A(\alpha_\varepsilon t)$, where $\alpha_\varepsilon = (1-\varepsilon)\rho^{-1}$.Then the traffic coefficient for system S^ε is equal to $1-\varepsilon$. Let $F_i^\varepsilon(x) = \lim \mathsf{P}(q_i^\varepsilon \leqslant x)$, $i = \overline{1, r}$ and $F^\varepsilon(x) = \lim \mathsf{P}(Q^\varepsilon \leqslant x)$, $i = \overline{1, r}$.

Theorem 3. If $\mathsf{E}\tau_j^{2+\delta} < \infty$, $\mathsf{E}\xi_j^{2+\delta} < \infty$, $\mathsf{E}\eta_j^{2+\delta} < \infty$ then in time-compression asymptotics for the choice rules R_0 and R_1

$$1 - F^\varepsilon(x/\varepsilon) \to \exp(-2x/\sigma_Q^2),$$

for choice rule R_2

$$1 - F_j^\varepsilon(x/\varepsilon) \to \exp(-2x/\sigma_2^2),\ \sigma_2^2 = \frac{\sigma_\eta^2}{b^2} + \frac{r-1}{r} + \frac{a^{-1}\sigma_\xi^2 + \mu^{-2}a\sigma_\tau^2 - 2\mu^{-1}\mathsf{cov}(\tau_1, \xi_1)}{r},$$

for a renewal $A(t)$ and choice rule R_3

$$1 - F_j^\varepsilon(x/\varepsilon) \to \exp(-2x/\sigma_3^2),\ \sigma_3^2 = \frac{\sigma_\eta^2}{b^2} + \frac{\sigma_\tau^2}{r\mu^2}, \quad as\ \varepsilon \to 0.$$

Corollary 2. If $A(t)$ is a renewal process then for choice rules R_0 and R_1

$$\mathsf{E}Q^\varepsilon \sim \frac{1}{2b^2\varepsilon}\left(\sigma_\eta^2 + \frac{b^2\sigma_\tau^2}{\mu^2}\right), \quad \text{as } \varepsilon \to 0,$$

for choice rule R_2

$$\mathsf{E}Q^\varepsilon \sim \frac{r}{2b^2\varepsilon}\left(\sigma_\eta^2 + \frac{r-1}{r}b^2 + \frac{b^2\sigma_\tau^2}{r\mu^2}\right), \quad \text{as } \varepsilon \to 0,$$

for choice rule R_3

$$\mathsf{E}Q^\varepsilon \sim \frac{r}{2b^2\varepsilon}\left(\sigma_\eta^2 + \frac{b^2\sigma_\tau^2}{r\mu^2}\right), \quad \text{as } \varepsilon \to 0.$$

Acknowledgements. Work is partially supported by RFBR grant 10-01-00266-a.

Queueing systems with Regenerative Input Flow and Heterogeneous Servers [1]

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Model description.

We consider queueing systems with r independent channels. The servers are indexed from 1 to r. Service times of customers are independent random variables. The service cumulative function of a customer which is assigned to the i-th server is $B_i(x)$ with finite expectation β_i^{-1} and Laplace-Stieltjes transformation $\beta_i(s)$. Arriving customers form the single queue and are served in the order of their arrival. A waiting customer is assigned to the first available server which has the lowest index.

The input flow $X(t)$ is assumed to be regenerative. We use the notation: θ_i are regenerative points of $X(t)$, $\tau_i = \theta_i - \theta_{i-1}$, $(i = 1, 2, \ldots)$ are regenerative periods, $\xi_i = X(\theta_i) - X(\theta_{i-1})$ is the number of customers arrived during i-th regenerative period. Assume that $a = E\xi_i < \infty$, $\tau = E\tau_i < \infty$. Define the arrival intensity λ of the process $X(t)$ as follows: $\lambda = \lim_{t\to\infty} \frac{X(t)}{t}$ *a.s.* From the strong law of large numbers we can conclude that $\lambda = a\tau^{-1}$.

Main results.

Consider a stochastic process $\mathbf{W}(t) = (W_1(t), \ldots, W_m(t))$ where $W_i(t)$ is the remaining time for server i which is required to serve all the customers arrived before time t and assigned to the i-th server.

$\mathbf{W}(t)$ is a regenarative stochastic process with regenerative points θ_i that satisfy the relation $\mathbf{W}(\theta_i - 0) = 0$. Under some conditions the process $\mathbf{W}(t)$ reaches zero state from any bounded set.

Definition 1. Stochastic process $\mathbf{W}(t)$ is ergodic if there exists

$$\lim_{t\to\infty} P\{\mathbf{W}(t) < \mathbf{y}\} = F(\mathbf{y}),$$

where $F(y)$ is independent from the initial distribution $W(0)$ and $F(\mathbf{y})$ is r-dimensional cumulative distribution function.

Definition 2. The process $\mathbf{W}(t)$ is stochastically bounded if $\forall\, \varepsilon > 0\ \exists\, \mathbf{y} = (y_1, \ldots, y_r)$, $\exists\, t_0$ such that for $t > t_0$

$$P\{\mathbf{W}(t) \leq \mathbf{y}\} > 1 - \varepsilon$$

Theorem 1. Process $\mathbf{W}(t)$ is ergodic iff it is stochastically bounded. Moreover, if $\mathbf{W}(t)$ is not ergodic then $\forall\, \varepsilon > 0$, $\forall\, \mathbf{y} = (y_1, \ldots, y_r)$, $\exists\, t_0$ such that for $t > t_0$

$$P\{\mathbf{W}(t) \geq \mathbf{y}\} > 1 - \varepsilon$$

Theorem 2. Under conditions $P\{\xi_1 \le 1,\ \eta_1^i < \tau_1\} > 0,\ \forall\, i \le r$

1. $\mathbf{W}(t)$ is ergodic if
$$\rho = \frac{\lambda}{\sum_{i=1}^{r} \beta_i} < 1.$$

2. If $\rho = 1$ and $E\tau_i^{2+\delta} < \infty, E\xi_i^{2+\gamma} < \infty$ for some $\delta > 0,\ \gamma > 0$ then $\mathbf{W}(t)$ is not ergodic.

3. If $\rho > 1$, then $\mathbf{W}(t)$ is not ergodic.

Unreliable servers.

Assume that the servers in the system described above are unreliable. We suppose that servers can fail only when they are occupied (i.e. serve a customer). Periods of functioning of the i-th server are exponentially distributed with the mean value γ_i^{-1}. Repair periods are independent of one another and of periods of functioning and for the i-th server have distribution function $R_i(x)$ with the mean value r_i. We make two different assumptions about what happens if a server fails while serving a customer:

1. the customer leaves the system immediately after interruption;

2. the customer's service is interrupted but will be continued from the beginning by the same server

Theorem 3. Theorem 2 holds true if $\rho = \frac{\lambda}{\sum_{i=1}^{r} \mu_i}$ where

1. for the first model case we suppose $\mu_i^{-1} = (r_i + \gamma_i^{-1})[1 - \beta_i(\gamma_i)]$

2. for the second model case we suppose $\mu_i^{-1} = \frac{(r_i+\gamma_i^{-1})[1-\beta_i(\gamma_i)]}{\beta_i(\gamma_i)}$

References

1. L. Afanasyeva, E. Bashtova, E. Bulinskaya *Limit Theorems for Semi-Markov Queues and Their Applications.* Communications in Statistics - Simulation and Computation, vol. 41, 2012, 12c.

2. J. Kiefer, J. Wolfowitz, *On the Theory of Queues with Many Servers*, Transactions of the American Mathematical Society Vol. 78, 1955.

3. W. Whitt, D. Iglehart, *Multiple Channel in Heavy Traffic I.* Adv. Appl. Prob. 2, 1970.

4. E. Morozov, *The stability of non-homogeneous queueing systems with regenerative input.* Preprint No R-95-2021, Dept. Math. and Computer Sci., Aalborg Univ., Aalborg, Denmark

5. E. Morozov, *Stochastic boundness of some queueing processes.* Preprint No R-95-2022, Dept. Math. and Computer Sci., Aalborg Univ., Aalborg.

Transfer theorems concerning asymptotic expansions for the distribution functions of statistics constructed from samples with random sizes

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Consider random variables (r.v.'s) $N_1, N_2, \ldots$ and $X_1, X_2, \ldots$, defined on the same probability space $(\Omega,\ \mathcal{A},\ \mathsf{P})$. By $X_1, X_2, \ldots X_n$ we will mean statistical observations whereas the r.v. N_n will be regarded as the random sample size depending on the parameter $n \in \mathbb{N}$. Assume that for each $n \geq 1$ the r.v. N_n takes only natural values (i.e., $N_n \in \mathbb{N}$) and is independent of the sequence $X_1, X_2, \ldots$ Everywhere in what follows the r.v.'s $X_1, X_2, \ldots$ are assumed independent and identically distributed.

For every $n \geq 1$ by $T_n = T_n(X_1, \ldots, X_n)$ denote a statistic, i.e., a real-valued measurable function of $X_1, \ldots, X_n$. For each $n \geq 1$ we define a r.v. T_{N_n} by setting $T_{N_n}(\omega) \equiv T_{N_n(\omega)}(X_1(\omega), \ldots, X_{N_n(\omega)}(\omega))$, $\omega \in \Omega$.

The following condition determines the asymptotic expansion (a.e.) for the distribution function (d.f.) of T_n with a non-random sample size.

Condition 1. *There exist $l \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\sigma > 0$, $\alpha > l/2$, $\gamma \geqslant 0$, $C_1 > 0$, a differentiable d.f. $F(x)$ and differentiable bounded functions $f_j(x), j = 1, \ldots, l$ such that*

$$\sup_x \Big| \mathsf{P}\big(\sigma n^{\gamma}(T_n - \mu) < x\big) - \ F(x) - \sum_{j=1}^{l} n^{-j/2} f_j(x) \Big| \leq \frac{C_1}{n^{\alpha}}, \quad n \in \mathbb{N}.$$

The following condition determines the a.e. for the d.f. of the normalized random index N_n.

Condition 2. *There exist $m \in \mathbb{N}$, $\beta > m/2$, $C_2 > 0$, a function $0 < g(n) \uparrow \infty, n \to \infty$, a d.f. $H(x), H(0+) = 0$ and functions $h_i(x)$, $i = 1, \ldots, m$ with bounded variation such that*

$$\sup_{x \geqslant 0} \Big| \mathsf{P}\Big(\frac{N_n}{g(n)} < x\Big) - H(x) \ - \sum_{i=1}^{m} n^{-i/2} h_i(x) \Big| \leq \frac{C_2}{n^{\beta}}, \quad n \in \mathbb{N}.$$

Define the function $G_n(x)$ as

$$G_n(x) = \int\limits_{1/g(n)}^{\infty} F(xy^\gamma)dH(y) + \sum_{i=1}^{m} n^{-i/2} \int\limits_{1/g(n)}^{\infty} F(xy^\gamma)dh_i(y) +$$

$$+\sum_{j=1}^{l} g^{-j/2}(n) \int\limits_{1/g(n)}^{\infty} y^{-j/2} f_j(xy^\gamma)dH(y)+$$

$$+\sum_{j=1}^{l}\sum_{i=1}^{m} n^{-i/2} g^{-j/2}(n) \int\limits_{1/g(n)}^{\infty} y^{-j/2} f_j(xy^\gamma)dh_i(y). \tag{1}$$

Theorem 1. *Let the statistic $T_n = T_n(X_1, ..., X_n)$ satisfy condition* 1 *and the r.v. N_n satisfy condition* 2*. Then there exists a constant $C_3 > 0$ such that*

$$\sup_x \left|\mathsf{P}\big(\sigma g^\gamma(n)(T_{N_n} - \mu) < x\big) - G_n(x)\right| \le C_1 \mathsf{E} N_n^{-\alpha} + \frac{C_3 + C_2 M_n}{n^\beta},$$

$$M_n = \sup_x \int\limits_{1/g(n)}^{\infty} \left|\frac{\partial}{\partial y}\Big(F(xy^\gamma) + \sum_{j=1}^{l} (yg(n))^{-j/2}\ f_j(xy^\gamma)\Big)\right| dy$$

and the function $G_n(x)$ is defined by (1)*.*

Let $\Phi(x)$ and $\varphi(x)$ respectively denote the d.f. of the standard normal law and its density.

Lemma 1. *Let $l = 1$, $0 < g(n) \uparrow \infty$, $F(x) = \Phi(x)$, $f_1(x) = \frac{1}{6}\mu_3\sigma^3(1 - x^2)\varphi(x)$. Then the quantity M_n in theorem* 1 *satisfies the inequality $M_n \leqslant 2 + \widetilde{C}|\mu_3|\sigma^3$, where*

$$\widetilde{C} = \frac{1}{3}\sup_{u \geqslant 0}\big\{\varphi(u)(u^4 + 2u^2 + 1)\big\} = \frac{16}{3\sqrt{2\pi e^3}} \approx 0.474752293191785...$$

Consider some examples of application of theorem 1.

Student distribution. Let $X_1, X_2, ...$ be i.i.d. r.v.'s with $\mathsf{E}X_1 = \mu$, $0 < \mathsf{D}X_1 = \sigma^{-2}$, $\mathsf{E}|X_1|^{3+2\delta} < \infty$, $\delta \in (0, \frac{1}{2})$ and $\mathsf{E}(X_1 - \mu)^3 = \mu_3$. For each n let

$$T_n = \tfrac{1}{n}(X_1 + ... + X_n). \tag{2}$$

Assume that the r.v. X_1 satisfies the Cramér condition (C)

$$\limsup_{|t|\to\infty} |\mathsf{E}\exp\{itX_1\}| < 1.$$

Let $G_\nu(x)$ be the Student d.f. with parameter $\nu > 0$ corresponding to the density

$$p_\nu(x) = \frac{\Gamma(\nu+1/2)}{\sqrt{\pi\nu}\Gamma(\nu/2)}\Big(1+\frac{x^2}{\nu}\Big)^{-(\nu+1)/2}, \quad x \in \mathbb{R},$$

where $\Gamma(\cdot)$ is the Euler's gamma-function and $\nu > 0$ is the shape parameter (if $\nu \in \mathbb{N}$, then ν is called *the number of degrees of freedom*). In practice, it can be arbitrarily small determining the typical heavy-tailed distribution. If $\nu = 2$, then the d.f. $G_2(x)$ is expressed explicitly as

$$G_2(x) = \frac{1}{2}\Big(1+\frac{x}{\sqrt{2+x^2}}\Big), \quad x \in \mathbb{R}.$$

for $\nu = 1$ we have the Cauchy distribution.

For $r > 0$ let

$$H_r(x) = \frac{r^r}{\Gamma(r)}\int_0^x e^{-ry}y^{r-1}dy, \quad x \geqslant 0,$$

be the gamma-d.f. with parameter $r > 0$. Denote

$$g_r(x) = \int_0^\infty \varphi(x\sqrt{y})\frac{1-x^2y}{\sqrt{y}}dH_r(y), \quad x \geqslant 0. \tag{3}$$

Theorem 2. *Let the statistic T_n have the form* (2)*, where $X_1, X_2, \ldots$ are i.i.d. r.v.'s with $\mathsf{E}X_1 = \mu$, $0 < \mathsf{D}X_1 = \sigma^{-2}$, $\mathsf{E}|X_1|^{3+2\delta} < \infty$, $\delta \in (0, \frac{1}{2})$ and $\mathsf{E}(X_1-\mu)^3 = \mu_3$. Moreover, assume that the r.v. X_1 satisfies the Cramér condition (C). Assume that for some $r > 0$ the r.v. N_n has the negative binomial distribution*

$$\mathsf{P}(N_n = k) = \frac{(k+r-2)\cdots r}{(k-1)!}\frac{1}{n^r}\Big(1-\frac{1}{n}\Big)^{k-1}, \quad k \in \mathbb{N}.$$

Let $G_{2r}(x)$ be the Student d.f. with parameter $\nu = 2r$ and $g_r(x)$ be defined by (3)*. Then for $r > 1/(1+2\delta)$, as $n \to \infty$, we have*

$$\sup_x\Big|\mathsf{P}\Big(\sigma\sqrt{r(n-1)+1}(T_{N_n}-\mu) < x\Big) - G_{2r}(x) - \frac{\mu_3\sigma^3 g_r(x)}{6\sqrt{r(n-1)+1}}\Big| =$$

$$= \begin{cases} O\Big(\Big(\frac{\log n}{n}\Big)^{1/2+\delta}\Big), & r = 1, \\ O\big(n^{-\min(1, r(1/2+\delta))}\big), & r > 1, \\ O\big(n^{-r(1/2+\delta)}\big), & (1+2\delta)^{-1} < r < 1. \end{cases}$$

Laplace distribution. Consider the Laplace d.f. $\Lambda_\theta(x)$ corresponding to the density

$$\lambda_\theta(x)=\frac{1}{\theta\sqrt{2}}\exp\Big\{-\frac{\sqrt{2}|x|}{\theta}\Big\},\quad \theta>0,\ x\in\mathbb{R}.$$

Let $Y_1,Y_2,\dots$ be i.i.d. r.v.'s with a continuous d.f. Set

$$N(s)=\min\Big\{i\geq 1:\ \max_{1\leq j\leq s}Y_j<\max_{s+1\leq k\leq s+i}Y_k\Big\}.$$

It is known that

$$\mathsf{P}(N(s)\geq k)=\frac{s}{s+k-1},\quad k\geq\ 1 \tag{4}$$

(see, e.g., [1] or [2]). Now let $N^{(1)}(s),N^{(2)}(s),\dots$ be i.i.d. r.v.'s distributed in accordance with (4). Define the r.v.

$$N_n(s)=\max_{1\leq j\leq n}N^{(j)}(s),$$

then, as it was shown in [3],

$$\lim_{n\to\infty}\mathsf{P}\Big(\frac{N_n(s)}{n}<x\Big)=e^{-s/x},\quad x>0,$$

and for an asymptotically normal statistic T_n we have

$$\mathsf{P}\big(\sigma\sqrt{n}(T_{N_n(s)}-\mu)<x\big)\longrightarrow\Lambda_{1/s}(x),\quad n\to\infty,\ x\in\mathbb{R},$$

where $\Lambda_{1/s}(x)$ is the Laplace d.f. with parameter $\theta=1/s$.

Denote

$$l_s(x)=\int\limits_0^\infty\varphi(x\sqrt{y})\frac{1-x^2y}{\sqrt{y}}\ de^{-s/y},\quad x\in\mathbb{R}. \tag{5}$$

Theorem 3. *Let the statistic T_n have the form (2), where $X_1,X_2,\dots$ are i.i.d. r.v.'s with $\mathsf{E}X_1=\mu$, $0<\mathsf{D}X_1=\sigma^{-2}$, $\mathsf{E}|X_1|^{3+2\delta}<\infty$, $\delta\in(0,\frac{1}{2})$ and $\mathsf{E}(X_1-\mu)^3=\mu_3$. Moreover, assume that the r.v. X_1 satisfies the Cramér condition (C). Assume that for some $s\in\mathbb{N}$ the r.v. $N_n(s)$ has the distribution*

$$\mathsf{P}(N_n(s)=k)=\Big(\frac{k}{s+k}\Big)^n-\Big(\frac{k-1}{s+k-1}\Big)^n,\quad k\in\mathbb{N}.$$

Then

$$\sup_x\Big|\mathsf{P}\big(\sigma\sqrt{n}(T_{N_n(s)}-\mu)<x\big)-\Lambda_{1/s}(x)-\frac{\mu_3\sigma^3l_s(x)}{6\sqrt{n}}\Big|=O\Big(\frac{1}{n^{1/2+\delta}}\Big),\quad n\to\infty,$$

where $l_s(x)$ is defined in (5).

This research was supported by the Russian Foundation for Basic Research (projects 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a), and by the Ministry for Education and Science of Russian Federation (state contract 16.740.11.0133).

References

1. *Wilks S. S.* Recurrence of extreme observations // Journal of American Mathematical Society, 1959. Vol. 1, No. 1, P. 106–112.

2. *Nevzorov V. B.* Records. Mathematical Theory. – Moscow: Fazis, 2000 (in Russian).

3. *Bening V. E., Korolev V. Yu.* Some statistical problems related to the Laplace distribution // Informatics and its Applications, 2008. Vol. 2. No. 2. P. 19–34.

On concentration functions of regular statistics constructed from samples with random sizes

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The concentration function (c.f.) of a random variable (r.v.) Z is defined as

$$Q_Z(\lambda) = \sup_{x\in\mathbb{R}} \mathsf{P}(x \leqslant Z \leqslant x+\lambda), \quad \lambda \geqslant 0,$$

see, e.g., [1]).

Lemma 1. *Let ξ and η be two r.v.'s. Then*

$$\sup_{\lambda\geqslant 0} |Q_\xi(\lambda) - Q_\eta(\lambda)| \leqslant 4 \sup_{x\in\mathbb{R}} |\mathsf{P}(\xi < x) - \mathsf{P}(\eta < x)|.$$

Lemma 2. *Let ξ be a r.v. with symmetric unimodal distribution. Then for $\lambda > 0$*

$$Q_\xi(\lambda) = \mathsf{P}\Big(|\xi| < \frac{\lambda}{2}\Big).$$

Consider random variables (r.v.'s) $N_1, N_2, \ldots$ and $X_1, X_2, \ldots$, defined on the same probability space $(\Omega, \mathcal{A}, \mathsf{P})$. By $X_1, X_2, \ldots X_n$ we will mean statistical observations whereas the r.v. N_n will be regarded as the random sample size depending on the parameter $n \in \mathbb{N}$. Assume that for each $n \geq 1$ the r.v. N_n takes only natural values (i.e., $N_n \in \mathbb{N}$) and is independent of the sequence $X_1, X_2, \ldots$ Everywhere in what follows the r.v.'s $X_1, X_2, \ldots$ are assumed independent and identically distributed.

For every $n \geq 1$ by $T_n = T_n(X_1, \ldots, X_n)$ denote a statistic, i.e., a real-valued measurable function of $X_1, \ldots, X_n$. For each $n \geq 1$ we define a r.v. T_{N_n} by setting $T_{N_n}(\omega) \equiv T_{N_n(\omega)}(X_1(\omega), \ldots, X_{N_n(\omega)}(\omega))$, $\omega \in \Omega$.

From lemmas 1 and 2 we obtain

THEOREM 1. *Assume that for some $\mu \in \mathbb{R}$, $C > 0$, $\sigma > 0$, $\nu \in \mathbb{R}$ and symmetric unimodal d.f. $G(x)$ the statistic T_{N_n} satisfies the inequality*

$$\sup_{x \in \mathbb{R}} \left| \mathsf{P}\big(\sigma n^{\delta}(T_{N_n} - \mu) < x\big) - G(x) \right| \leqslant \frac{C}{n^{\gamma}}.$$

Then for any $n \in \mathbb{N}$ we have

$$\sup_{\lambda \geqslant 0} \left| Q_{T_{N_n}}(\lambda) - 2G\Big(\frac{\lambda \sigma n^{\delta}}{2}\Big) + 1 \right| \leqslant \frac{4C}{n^{\gamma}}.$$

Let $G_{\nu}(x)$ be the Student d.f. with parameter $\nu > 0$ corresponding to the density

$$p_{\nu}(x) = \frac{\Gamma(\nu + 1/2)}{\sqrt{\pi\nu}\,\Gamma(\nu/2)} \Big(1 + \frac{x^2}{\nu}\Big)^{-(\nu+1)/2}, \quad x \in \mathbb{R},$$

where $\Gamma(\cdot)$ is the Euler's gamma-function and $\nu > 0$ is the shape parameter (if $\nu \in \mathbb{N}$, then ν is called *the number of degrees of freedom*). In practice, it can be arbitrarily small determining the typical heavy-tailed distribution. If $\nu = 2$, then the d.f. $G_2(x)$ is expressed explicitly as

$$G_2(x) = \frac{1}{2}\Big(1 + \frac{x}{\sqrt{2 + x^2}}\Big), \quad x \in \mathbb{R}.$$

for $\nu = 1$ we have the Cauchy distribution.

Assume that a statistic T_n is asymptotically normal so that

$$\sup_{x \in \mathbb{R}} \left| \mathsf{P}\Big(\sigma\sqrt{n}(T_n - \mu) < x\Big) - \Phi(x) \right| \leqslant \frac{C_0}{\sqrt{n}}, \quad n \in \mathbb{N}, \tag{1}$$

where the quantity $C_0 > 0$ does not depend on n.

From theorem 1 and the results of [2] and [3] we obtain

THEOREM 2. *Assume that for some $\mu \in \mathbb{R}$, $C_0 > 0$, $\sigma > 0$ the statistic T_n satisfies* (1). *Assume also that for some $r > 0$ the r.v. N_n has the negative binomial distribution*

$$\mathsf{P}(N_n = k) = \frac{(k + r - 2) \cdots r}{(k-1)!} \frac{1}{n^r} \Big(1 - \frac{1}{n}\Big)^{k-1}, \quad k \in \mathbb{N}.$$

Then for $r \in (0, \frac{1}{2})$ and any $n \in \mathbb{N}$

$$\sup_{\lambda \geqslant 0} \Big| Q_{T_{N_n}}(\lambda) - 2G_{2r}\Big(\frac{\lambda\sigma\sqrt{rn}}{2}\Big) + 1 \Big| \leqslant \frac{4C_1}{n^r}.$$

If $r = \frac{1}{2}$, then

$$\sup_{\lambda \geqslant 0} \Big| Q_{T_{N_n}}(\lambda) - \frac{2}{\pi}\arctan\Big(\frac{\lambda\sigma\sqrt{n}}{2\sqrt{2}}\Big) \Big| \leqslant 4C_2\frac{\log n}{\sqrt{n}}, \quad n > 1.$$

If $r > \frac{1}{2}$, then

$$\sup_{\lambda \geqslant 0} \Big| Q_{T_{N_n}}(\lambda) - 2G_{2r}\Big(\frac{\lambda\sigma\sqrt{rn}}{2}\Big) + 1 \Big| \leqslant \frac{4C_3}{\sqrt{n}},$$

Here $G_{2r}(x)$ is the Student d.f. with parameter $\nu = 2r$ and $C_1 = C_1(r)$, C_2, $C_3 = C_3(r)$ do not depend on n. In particular, if $r = 1$, that is, if the r.v. N_n has the geometric distribution with parameter $\frac{1}{n}$, then

$$\sup_{\lambda \geqslant 0} \Big| Q_{T_{N_n}}(\lambda) - \frac{\lambda\sigma\sqrt{n}}{\sqrt{8 + \lambda^2\sigma^2 n}} \Big| \leqslant \frac{4C_3}{\sqrt{n}}.$$

Consider the Laplace d.f. $\Lambda_\theta(x)$ corresponding to the density

$$\lambda_\theta(x) = \frac{1}{\theta\sqrt{2}}\exp\Big\{-\frac{\sqrt{2}|x|}{\theta}\Big\}, \quad \theta > 0,\ x \in \mathbb{R}.$$

Let $Y_1, Y_2, ...$ be i.i.d. r.v.'s with a continuous d.f. Set

$$N(s) = \min\Big\{ i \geq 1 : \max_{1 \leq j \leq s} Y_j < \max_{s+1 \leq k \leq s+i} Y_k \Big\}.$$

It is known that

$$\mathsf{P}(N(s) \geq k) = \frac{s}{s+k-1}, \quad k \geq 1 \tag{2}$$

(see, e.g., [4] or [5]). Now let $N^{(1)}(s), N^{(2)}(s), ...$ be i.i.d. r.v.'s distributed in accordance with (2). Define the r.v.

$$N_n(s) = \max_{1 \leq j \leq n} N^{(j)}(s), \tag{3}$$

then, as it was shown in [6], for an asymptotically normal statistic T_n we have

$$\mathsf{P}\big(\sigma\sqrt{n}(T_{N_n(s)} - \mu) < x\big) \longrightarrow \Lambda_{1/s}(x), \quad n \to \infty,\ x \in \mathbb{R},$$

where $\Lambda_{1/s}(x)$ is the Laplace d.f. with parameter $\theta = 1/s$. If the statistic T_n satisfies relation (1), then, as was shown in [7],

$$\sup_{x \in \mathbb{R}} \Big| \mathsf{P}\Big(\sigma\sqrt{n}(T_{N_n(m)} - \mu) < x\Big) - \Lambda_{1/m}(x) \Big| \leqslant \frac{C_4}{\sqrt{n}}, \quad n \in \mathbb{N},$$

where the quantity $C_4 = C_4(s)$ does not depend on n. This inequality and theorem 1 imply the following result.

THEOREM 3. *Assume that for some $\mu \in \mathbb{R}$, $C_0 > 0$, $\sigma > 0$ the statistic T_n satisfies* (1) *and for an $s \in \mathbb{N}$ the r.v. $N_n(s)$ is defined by* (3). *Then for any $n \in \mathbb{N}$ we have*

$$\sup_{\lambda \geqslant 0} \left| Q_{T_{N_n(s)}}(\lambda) - 2\Lambda_{1/s}\Big(\frac{\lambda\sigma\sqrt{n}}{2}\Big) + 1 \right| \leqslant \frac{4C_4}{\sqrt{n}}.$$

We also consider some particular examples dealing with U-, L- and R-statistics.

This research was supported by the Russian Foundation for Basic Research (projects 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a), and by the Ministry for Education and Science of Russian Federation (state contract 16.740.11.0133).

References

1. *Hengartner W., Theodorescu R.* Concentration Functions. – New York - London: Academic Press, 1973.

2. *Gavrilenko S. V., Zubov V. N., Korolev V. Yu.* An estimate of the rate of convergence of the distributions of regular statistics constructed from samples with random sizes with negative binomial distribution to the Student distribution // Statistical Methods of Estimation and Testing Hypotheses. – Perm: Rerm State University Publishing House, 2006. P. 118–134 (in Russian).

3. *Nefedova Yu. S.* Estimates of the rate of convergence in the limit theorem for negative binomial random sums // Statistical Methods of Estimation and Testing Hypotheses. – Perm: Rerm State University Publishing House, 2011. P. 46–61 (in Russian).

4. *Wilks S. S.* Recurrence of extreme observations // Journal of American Mathematical Society, 1959. Vol. 1, No. 1, P. 106–112.

5. *Nevzorov V. B.* Records. Mathematical Theory. – Moscow: Fazis, 2000 (in Russian).

6. *Bening V. E., Korolev V. Yu.* Some statistical problems related to the Laplace distribution // Informatics and its Applications, 2008. Vol. 2. No. 2. P. 19–34.

7. *Lyamin O. O.* On the rate of convergence of the distributions of some statistics to the Laplace and Student distributions // Bulletin of Moscow University. Ser. 15 Computational Math. and Cybern., 2011. No. 1. P. 39–47.

Optimal control and stability of some inventory and insurance models

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At first we consider stochastic models of systems with several replenishment sources, arising in inventory, insurance and other applications. Our aim is to establish the optimal control providing maximum (or minimum) of various objective functions. To this end (see, e.g., Bulinskaya [1]) we study the asymptotic behavior of underlying discrete- and continuous-time processes. For illustration, we formulate below some results for the discrete-time two-supplier inventory model which can be considered as modification of that treated in Bulinskaya [2].

Let $f_n(x)$ be the minimal n-step expected costs if x is the initial inventory level. We denote by c_i the order cost of a unit delivered by supplier i, $i = 1, 2$, h the holding cost, r the deficit penalty, α being the discount factor. Then the following Bellman equation is valid for $n \geqslant 1$

$$f_n(x) = -c_1 x + \min_{u \geqslant v \geqslant x} G_n(u, v), \quad f_0(x) \equiv 0,$$

$$G_n(u, v) = c_2 u + (c_1 - c_2)v + pL(u) + qL(v) + \alpha \mathsf{E} f_{n-1}(u - \xi_1).$$

Here p is probability that the second (unreliable) supplier delivers order immediately and $q = 1 - p$ the probability of one-period delay, whereas $L(u) = \mathsf{E}[h(u - \xi_1)^+ + r(\xi_1 - u)^+]$ gives mean costs during the first period starting from the level u and ξ_1 is the inventory demand in this period. It is supposed that the sequence $\{\xi_k\}_{k \geqslant 1}$ consists of i.i.d. r.v.'s with a known distribution function and the first supplier delivers orders immediately.

It is proved that optimal policy has a threshold character. Moreover, in contrast with the case $p = 0$ considered previously, under some additional assumptions it is optimal to use the unreliable supplier even for $n = 1$.

The second model deals with functioning of insurance company under the following assumption. At the beginning of each period (usually year) it is possible either to invest some money amount or borrow. The aim is to find a decision minimizing the n-step expected costs.

We study several cases. The simplest assumption is the fixed premium c acquired each period, the rates of investing and borrowing equal to r. If the company capital is less than the demand of policyholders, an urgent loan can be obtained at the rate q, $q > r$. Let x be the initial capital and demand amounts in different periods form a sequence of i.i.d. r.v.'s with d.f. F.

Proposition. *Under the above assumptions optimal decision for each n is given by $y^*(x) = F^{-1}(1 - rq^{-1}) - x - c$.*

That means, the optimal policy is stationary. Moreover, if $y^*(x) > 0$ then it is necessary to borrow this amount. If $y^*(x) < 0$ the company invests $|y^*(x)|$.

In the second case not only the indemnity is supposed random but the premium amount as well. If the investment and borrowing rates are the same the optimal decision is independent of n. On the contrary, for different rates the optimal decision depends on n.

Hence, another direction of our investigation is construction of stationary asymptotically optimal policy in cases of known and unknown demand distributions.

The study of the models stability to small fluctuations of parameters and perturbations of underlying probability distributions is carried out along the same lines as in Bulinskaya [3].

We apply not only the cost approach but a reliability one as well. However, instead of the usual ruin probability studied in classical models, in the case of borrowing it is appropriate to use the notion of absolute ruin (see, e.g., [4]).

Some numerical examples are also provided.

Acknowledgement. The research was partially supported by RFBR grant 10-01-00266.

References

1. E. Bulinskaya. Some aspects of decision making under uncertainty. *Journal of Statistical Planning and Inference*, 2007, vol. 147, p. 2613–2632.

2. E. Bulinskaya. Stochastic Insurance Models. Their Optimality and Stability. *C.H.Skiadas (ed.) Advances in Data Analysis*, 2010, Birkhäuser, Boston, p. 129–140.

3. E. Bulinskaya. Sensitivity analysis of some applied models. *Pliska Studia Mathematica Bulgarica*, 2007, vol. 18, p. 57–90.

4. I-R. Mitric, A.L. Badescu, D.A. Stanford. On the absolute ruin problem in a Sparre Andersen risk model with constant interest. *Insurance: Mathematics and Economics*, 2012, vol. 50, p. 167–178.

Stochastic characteristics of magnetoencephalogram and myogramm signals

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The brain functional mapping task was set at the Department of Mathematical Statistics, Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University. We developed and analysed a mathematical model of magnetoencephalogram and myogram signals. We described the main characteristics of these signals. Our findings underlay further signal processing and localization algorithm development, including effective noise filter and reduction.

We demonstrated non-stationarity of these signals. Hence, averaging of the data characteristics shown not to be reasonable.

Regarding localization problems the assumption of noise normality can either greatly simplify the model, or distort it. Therefore, we studied the noise empirical distribution. To test noise normality assumption (null hypothesis) we engaged Chi-square test with significance level α set at 0.05. For sample size of 76000 the resulting Chi-square statistic was $1.275 \cdot 10^3$ while rejection limit was 14.1. So we concluded that null hypothesis was not consistent with empirical data.

Similarly normality assumption with regard to myogram noise was tested. In this case resulting statistic was $3.5855 \cdot 10^4$ and null hypothesis was rejected.

Median equality hypothesis for several samples was also rejected. The samples assumed to be non-normal, Kruskal-Wallis analysis demonstrated that the samples have significantly different medians.

Our findings demonstrated extreme complexity and specific nature of studied biomedical signals.

References

1. *Zakharova T.V., Nikiforov S.Y., Goncharenko M.B., Dranitsyna M.A., Klimov G.A., Khaziakhmetov M.S., Chayanov N.V.* Signal processing methods for the localization of nonrenewable brain regions // Informatics and Applications(article in press).

2. *Zakharova T.V., Shestakov O.V.* Wavelet analysis and its applications: tutorial. 2nd ed. – M: INFRA-M, 2012.

3. *Hamalainen M., Hari R., Ilmoniemi R. J., Knuutila J., Lounasmaa O. V.* Magnetoencephalography – theory, instrumentation, and applications to noninvasive studies of the working human brain // Rev. Mod. Phys., 1993. Vol. 65. P. 413-497.

Asymptotic distribution of smoothness estimator in Besov spaces

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In the paper J. Nonparametr. Stat.23 (4) (2011), Dziedziul, Kucharska, Wolnik define a parameter of smoothness of function. The parameter is given in terms of a Besov spaces. They construct and examine properties of estimator of the parameter of smoothness of density.

Now we propose a modified estimator of the parameter of smoothness. Using this method we identify defects (singularities) of a density function, i.e we assume that the density function has compact support and it is partially smooth (smooth except a few points). By applying Daubechies's wavelets we find the singularities of density. To improve the rate of convergence of the estimator we propose the procedure of sample enrichment, i.e. we add the control sample from smoother density. Taking a sample from such new distribution we prove asymptotic formulas for this estimator. The proof is based on Berry Essen's inequality.

On max-compound Cox processes

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Max-compound Cox process are used as mathematical models of flows of catastrophic events insurance, financial mathematics, engineering, etc.

Let $X_1, X_2 \ldots$ be independent random variables with common distribution function $F(x)$. Let $N(t)$ be a Cox process controlled by a random measure $\Lambda(t)$. Define the max-compound Cox process as

$$M(t) = \begin{cases} -\infty, & \text{if } N(t) = 0, \\ \max\limits_{1 \le k \le N(t)} X_k, & \text{if } N(t) \ge 1, \end{cases} \quad t \ge 0.$$

For real functions $a(t)$ and $b(t) > 0$ denote

$$F_N(x,t) = P\left(\frac{1}{b(t)}\left(\max_{1 \le k \le N(t)} X_k - a(t)\right) < x\right)$$

In this work we describe necessary and sufficient conditions for weak convergence of $F_N(x,t)$ to some distribution function $H(x)$ when $t \to \infty$ and specific form of $H(x)$ as well. Also some results concerning the convergence rate are presented.

This research was supported by the Russian Foundation for Basic Research (project 12-01-16086-mob-z-ross).

References

1. V. Yu. Korolev. *Mathematical Models of Non-Homogeneous Flows of Extremal Events.* – Moscow: Torus Press, 2008 (in Russian).

2. V. Yu. Korolev, D. O. Selivanova. Convergence rate estimates in some limit theorems for maximum random sums, *J. of Math. Sci.*, 1995, vol. 76, no. 1, p. 2163–2168.

3. Ja. Galambos. *The Asymptotic Theory of Extreme Order Statistics.* – New York: Wiley, 1978.

4. M. R. Leadbetter, G. Lindgren, H. Rootzen. *Extremes and Related Properties of Random Sequences and Processes.* – Berlin, New York: Springer, 1986.

Positioning of the spacecraft during docking to the ISS

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Space flights and orbital experiments are an important part of todays science and national profile.

At the moment there is only space station. Since the station is permanently inhabited and to support its functioning and crew life and conduct scientific experiments reliable ferry and crew transport system is strongly required. Space flights and orbital experiments are an important part of modern science and national profile.

At the moment there is only space station. Since the station is permanently inhabited and to support its functioning and crew life and conduct scientific experiments reliable ferry and crew transport system is strongly required.

At the moment Russian spacecrafts Soyuz and Progress are the most reliable transport for the flights to the station, in particular due to the exclusive automatic rendezvous and docking system Course.

At present time all dockings are carried out in automatic mode. In case of some accident or Course system failure the spacecraft remains out of automatic control.

So the task was to develop backup positioning system for such accidental situations. This system is based on a fundamentally different principle of parameter calculation. It should work in parallel and provide equal accuracy (deviation of docking station central axis should not exceed 7 cm).

The input signal assumed to be received from spacecraft front camera (frame frequency is 25 per second). At the Department of Mathematical Statistics, Faculty of Computational Mathematics and Cybernetics, Moscow State University we work on developing of such system. Designed algorithm is based on the processing of various station nodes and using them as input parameters. Configuration of these nodes as well as their relative positions are documented and known before. This is a real-time algorithm and hence it incorporates the minimizing principle for required computations.

For instance, firstly preliminary computations are performed, then a certain area is localized, and after specified.

During a single frame processing the convolution with Gaussian derivatives in various directions is calculated to localize the edges of different directions with subsequent threshold processing (see [1, 2]) and build a map-edges for final space craft positioning. These derivatives are defined as:

$$\frac{\partial G(x,y,\sigma)}{\partial x} = -\frac{x}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}.$$

Vector field of displacement between key frames based on optical flow equation assumed to underlie the calculation of the relative motion:

$$\bigtriangledown I \cdot V + I_t = 0,$$

where $I(X,t)$ – image brightness at the point $X = (x,y)$ at the moment t, $V = \frac{\delta X}{\delta t}$. We work on parameter refinement and adjustment for better reliability.

Here is link on a demo-video:
`http://narod.ru/disk/50503495001.4ecfce02ab7b35a0454240fd51dce37f/for_demonstration_plus_ro_ax_ro_dot.avi.html`

The algorithm is tested on a special computer model of the ISS[1].

References

1. J.Canny. *A Computational Approach to Edge Detection.* Transactions on pattern analysis and machine intelligence, 1986, vol. PAMI-8, 6, p. 679-698.

2. T.V.Zakharova, O.V.Shestakov. *Wavelet analysis and its application.* Max Press, 2009 [in Russian].

The splitting component model for finite normal mixtures

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To investigate fine structure of compound process, stochastic models based on finite normal mixtures are used. The paper suggests practically important splitting component model for the finite scale and location mixtures. For each situation theorems about parameter stability in terms of inequalities for Levy distance between mixing distributions and Levy distance between mixtures are proved. The stability implies correspondence of the models with experimental data.

[1]The model developed by Department of Computational Mathematics and Cybernetics Lomonosov Moscow State University members and graduate students under the direction of V.V.Sazonov and S.B.Berezin

Consider *finite normal scale mixtures.* All observations X_i, $i = 1, \dots, k$, are independent and identically distributed and have cumulative distribution function

$$G(x) = \mathbb{E}\Phi(Ux) = \sum_{i=1}^{k} p_i \Phi(x\sigma_i), \sum_{i=1}^{k} p_i = 1,\ p_i \geqslant 0,\ \sigma_i > 0,\ i = 1, \dots, k,$$

where $\Phi(\cdot)$ is standard normal cumulative distribution function, U is discrete random variable taking value σ_i with probability p_i, $i = 1, \dots, k$.

In this case the splitting components model can be represented in following form. Assume that each observation X_i is independent and identically distributed and has cumulative distribution function

$$G_p(x) = \sum_{i=1}^{k-1} p_i \Phi(x\sigma_i) + (p_k - p)\Phi(x\sigma_k) + p\Phi(x\sigma), \quad \sigma > 0,\ 0 \leqslant p \leqslant p_k,$$

where all σ_i, p_i are known, σ and p are parameters of model. Assume without loss of generality $0 < \sigma_1 \leqslant \sigma_2 \leqslant \ldots \leqslant \sigma_{k-1} \leqslant \sigma \leqslant \sigma_k$.

U_p is random discrete variable taking value σ_i with probability p_i, $i = 1, \dots, k-1$, value σ with probability p, value σ_k with probability $p_k - p$. Lévy distance between U and U_p is represented in form $L(U, U_p) = \min\{\sigma_k - \sigma, p\}$.

Theorem 1. *In splitting component model for finite normal scale mixtures inequalities*

$$\frac{\sigma_1\sqrt{2\pi e}}{\max\{1, \sigma_k\}} L(G, G_p) \leqslant L(U, U_p) \leqslant \varphi^{-1/2}(\sigma_k) \left(1 + \frac{\sigma_k}{\sqrt{2\pi}}\right)^{1/2} L^{1/2}(G, G_p)$$

hold under the assumptions above.

Proof of Theorem 1 can be found in [1].

Consider *finite normal location mixtures.* All observations X_i, $i = 1, \dots, k$, are independent and identically distributed and have cumulative distribution function

$$F(x) = \mathbb{E}\Phi(x - V) = \sum_{i=1}^{k} p_i \Phi(x - a_i),\ \sum_{i=1}^{k} p_i = 1,\ p_i \geqslant 0,\ a_i \in \mathbb{R},\ i = 1, \dots, k,$$

where V is discrete random variable taking value a_i with probability p_i, $i = 1, \dots, k$.

In this case the splitting components model can be represented in following form. Assume that each observation X_i is independent and identically distributed and has cumulative distribution function

$$F_p(x) = \sum_{i=1}^{k-1} p_i \Phi(x - a_i) + (p_k - p)\Phi(x - a_k) + p\Phi(x - a), \quad a \in \mathbb{R},\ 0 \leqslant p \leqslant p_k,$$

where all a_i, p_i are known, a and p are parameters of model. Assume without loss of generality $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{k-1} \leqslant a \leqslant a_k$.

V_p is random discrete variable taking value a_i with probability p_i, $i = 1, \ldots, k-1$, value a with probability p and value a_k with probability $p_k - p$. Lévy distance between V and V_p is represented in form $L(V, V_p) = \min\{a_k - a, p\}$.

Theorem 2. *In splitting component model for finite normal location mixtures inequalities*

$$\frac{\sqrt{2\pi}}{\max\{1, a_k - a_{k-1}\}} L(F, F_p) \leqslant L(V, V_p) \leqslant \left(\frac{\left(1 + \frac{1}{\sqrt{2\pi}}\right) L(F, F_p)}{\varphi\Big(a_k + |a_k| - \min\{0, a_{k-1}\}\Big)} \right)^{1/2}$$

hold under the assumptions above.

Proof of Theorem 2 can be found in [2]. The results can be used for testing statistical hypotheses about the number of mixture components [3].

References

1. A. K. Gorshenin. Stability of normal scale mixtures with respect to variations in mixing distribution. *Systems and Means of Informatics*, 2012, vol. 22, No. 1, p. 136–148.

2. A. K. Gorshenin. On stability of normal location mixtures with respect to variations in mixing distribution. *Informatics and Applications*, 2012, vol. 6, No. 2 p. 22–28.

3. A. K. Gorshenin. Testing of statistical hypotheses in the splitting component model. *Moscow University Computational Mathematics and Cybernetics*, 2011, vol. 35, No. 4, p. 176–183.

Evolution of histograms and Fourier spectra in structural plasma turbulence in L-2M stellarator

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Fig. 1 shows the time evolution of the Fourier spectra of the low-frequency plasma fluctuations under different external conditions. It is known (see, for example, [1]) that these fluctuations are close to structural plasma turbulence due to the similarities of autocorrelation functions, values of moments and shapes of the probability densities. Development of new methods of analysis of

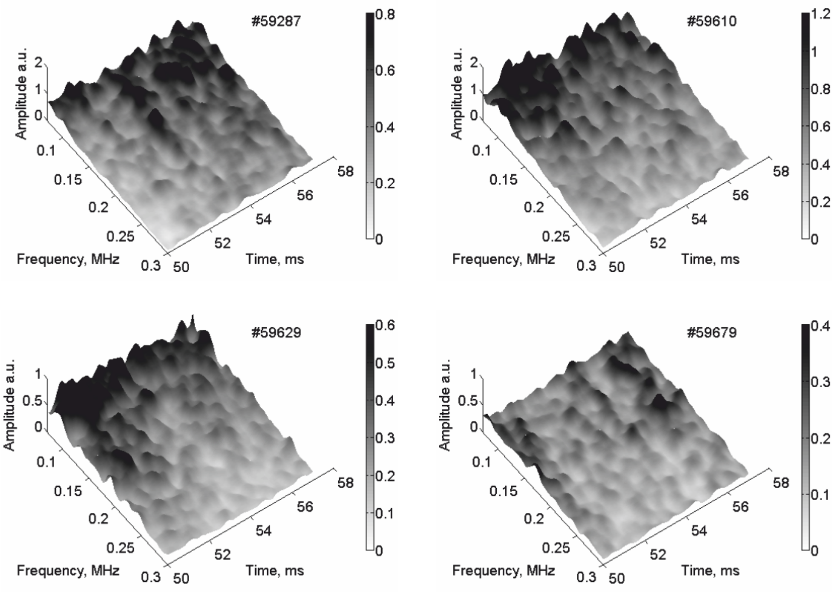

Figure 1: Spectrum decomposition.

plasma turbulence is very important in fields of plasma physics and controlled fusion.

Decomposition of the spectra into the components is discussed in [2]. Application of this approach for series of the experimental spectra gives opportunity to reveal components' evolution in plasma turbulence. It can help identify specific components which persist over time. With using the physical interpretation of the components it can be possible to create more precise models of the functioning of the plasma turbulence.

Acknowledgements. The work is supported by Russian Foundation for Basic Research, project 12-07-31267, and the Ministry for Science and Education of Russian Federation, state contract P770.

References

1. N. N. Skvortsova, D. K. Akulina, G. M. Batanov, N. K. Kharchev, L. V. Kolik, L. M. Kovrizhnykh, A. A. Letunov, V. P. Logvinenko, D. V. Malakhov, A. E. Petrov, A. A. Pshenichnikov, K. A. Sarksyan, G. S. Voronov. Effect of ECRH regime on characteristics of short-wave turbulence in plasma of the L-2M stellarator. *Plasma Phys. Control. Fusion*, 2010, vol. 52, No. 5, 11 pp.

2. A. K. Gorshenin, V. Yu. Korolev, D. V. Malakhov, N. N. Skvortsova. Analysis of fine stochastic structure of chaotic processes by kernel estimators. *Mathematical Modelling*, 2011, vol. 23, No. 4, p. 83–89.

Optimal investment for an Erlang(n) risk process

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Introduction. We consider an insurance company and a risky asset in which the company can invest the surplus.

Let T_i be the occurrence time of the i-th claim, N_t the number of claims in time interval $(0;t]$ and $Y_i > 0$ the amount of the i-th claim. Claim sizes Y_i are assumed i.i.d. Let $S_t = \sum_{i=1}^{N_t} Y_i$ denote the aggregate claim process; $s > 0$ is the initial surplus and $c > 0$ is the premium intensity of the insurer. Then the risk process $R_t = s + ct - S_t$.

We model the risk process of the insurance company as the Sparre-Andersen process with claim inter-arrival time distributed as Erlang(n) with scale parameter β, i.e. claim arrival times T_i are i.i.d. random variables with density function $q(x) = \beta^n x^{n-1} e^{-\beta x}/(n-1)!$.

The price Z_t of the risky asset is modelled by geometric Brownian motion with parameters μ and σ, i.e. $Z_t = \exp\{\sigma W_t + (\mu - 2^{-1}\sigma^2)t\}$, where W_t is a standard Brownian motion.

We assume that the company follows some investment strategy $\{A_t\}$, where A_t is the amount invested at time t into the risky asset. We assume that processes S and W are independent, and filtration $\{\mathcal{F}_t\}$ is generated by the Brownian motion $\{W_t\}$.

In described scenario our goal is to find the optimal investment strategy $\{A_t\}$ which maximizes the survival probability. In this paper we consider both finite and infinite time horizon. Let τ^A denote the ruin time of the company using strategy A_t, i.e. $\tau^A := \inf\{t \geq 0 : R_t^A < 0\}$. Then $\delta^A(s) = P[\tau^A = \infty | R_0^A = s]$ (in the finite horizon case $\delta_{\mathcal{T}}^A(s) = P[\tau^A > \mathcal{T} | R_0^A = s]$ for some $\mathcal{T} < \infty$) is the survival probability of the insurer using strategy $\{A_t\}$ with initial surplus s. We calculate the value function $\delta(s) = \sup_A\{\delta^A(s)\}$ to find optimal strategy $\{A_t^*\}$ where supremum is attained.

Hamilton-Jacobi-Bellmann equation Following the approach proposed in Dickson and Hipp [1], Hipp and Plum [2] or Schmidli [3] in this case it is also convenient to find optimal strategy by solving the corresponding Hamilton-Jacobi-Bellmann equation.

Suppose that the function $\delta(s)$ has the n-th derivative, stochastic integrals with respect to Brownian motion are martingales and all limits and expectations can be interchanged. For instance, we obtain the following equation for the optimal survival probability $\delta(s)$ in the infinite time horizon case

$$\sup_{A\geq 0}\left\{\left(-(c+\mu A)\frac{d}{ds}-\frac{\sigma^2 A}{2}\frac{d^2}{ds^2}+\beta\right)^n\delta(s)-\beta^n E[\delta(s-x)]\right\}=0.$$

In this paper we prove the existance of the optimal investment strategy in described cases and provide some numerical examples in order to illustrate the theory.

References

1. D. C. M. Dickson, C. Hipp. *Ruin probabilities for Erlang(2) risk processes*, 1997, Working Paper 148, Laboratory of Actuarial Mathematics, Copenhagen University.

2. C. Hipp, M. Plum. Optimal investment for insurers. *Insurance: Math. Econom.*, 2000, No 27, p. 215–228.

3. H. Schmidli. *Stochastic Control In Insurance*, 2008, Springer-Verlag: London.

Lower bounds for average observations number in selection and ranking of binomial and Poisson populations.

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Let we have m populations with identical distributions up to unknown parameter $\theta \in \Theta \subset R$. The parameter of i-th population will be denoted as θ_i. The study considers the procedures of selecting and ranking the populations with respect to their parameter values.

More specifically, we consider the selection procedures whose purpose is to select a population with the largest value of θ and the ranking procedures whose purpose is to rank the populations in ascending order of θ values. For given $0 < \Delta < 1$, let indifference zone be $\theta_{[m-1]}/\theta_{[m]} \leq 1-\Delta$ for selection procedures, and $\theta_{[i]}/\theta_{[i+1]} \leq 1-\Delta, \quad 0 \leq i \leq m-1$ for ranking procedures where $\theta_{[1]} \leq \cdots \leq \theta_{[m]}$. The procedures are required to have at least $1-\alpha$ probability of the correct decision whether the population parameters satisfy the indifference zone conditions.

The main question of the work is constructing lower bounds for the average sample size $\nu = \nu_1 + \cdots + \nu_m$ of selection and ranking procedures for binomial and Poisson population distributions. Lower bounds for general selection and ranking problems were obtained in articles [1] and [2] by using the universal lower bounds of Volodin [3] and Malyutov [4]. We apply the results of [1] and [2] to the parameters of the binomial and Poisson distributions.

Let $\omega(x, y) = x(\ln x - \ln(1-y)) + (1-x)(\ln(1-x) - \ln y)$ – Wald's function, $I(\theta, \vartheta)$ – Kullback-Leibler information divergence. Let us remind that the information divergence for the probability of success in Bernoulli trials is

$$I(\theta, \vartheta) = \theta \ln \frac{\theta(1-\vartheta)}{\vartheta(1-\theta)} + \ln \frac{1-\theta}{1-\vartheta}.$$

We assume for binomial distribution that $\theta_{[m]} < 1 - \Delta$. The information divergence for the parameter of Poisson distribution is

$$I(\theta, \vartheta) = \vartheta - \theta + \theta \ln \frac{\theta}{\vartheta}.$$

Then for both cases of population distribution and any selection procedure true the lower bound for the mean of sample size

$$E_\theta \nu \geq \omega(\alpha, \alpha) \sum_{i=1}^{m-1} I\left(\theta_{[i]}, \frac{1-t}{1-\Delta}\theta_{[m]}\right)^{-1}$$

where t is a root in interval $[0, 1 - \theta_{[m-1]}/\theta_{[m]}]$ of equation

$$\sum_{i=1}^{m-1} \frac{I(\theta_{[m]}, (1-t)\theta_{[m]})}{I(\theta_{[i]}, (1-t)\theta_{[m]}/(1-\Delta))} = 1.$$

For ranking procedures and $m \geq 3$ we have the lower bound

$$E_\theta \nu \geq \omega(\alpha, \alpha) \left(\frac{1}{I(\theta_{[1]}, \vartheta_1)} + \sum_{i=2}^{m-1} \frac{1}{2I(\theta_{[i]}, \vartheta_i)} \right)$$

where

$$\vartheta_1 = \min\{v_1(2),\ (1-\Delta)\theta_{[3]}\},$$
$$\vartheta_i = \min\{\max\{v_i(1),\ \theta_{[i-1]}/(1-\Delta)^2\},\ (1-\Delta)\theta_{[i+2]}\},$$
$$\vartheta_{m-1} = \max\{v_{m-1}(1/2),\ \theta_{[m-2]}/(1-\Delta)^2\}$$

and $v_i(c)$ with $c > 0$ is a value of ϑ such that

$$I(\theta_{[i]}, \vartheta) = cI(\theta_{[i+1]}, (1-\Delta)\vartheta), \quad \theta_{[i]} \leq \vartheta \leq \frac{\theta_{[i+1]}}{1-\Delta}.$$

The efficiency of well known selection and ranking procedures relatively obtained bounds is under investigation.

References

1. I. A. Kareev. *Lower Bounds for Average Sample Size and Efficiency of Sequential Selection Procedures.* Theory of Probability and its Applications, 2012, Vol. 56, No. 2. (to be appeared)

2. I. A. Kareev. *Lower Bounds for Average Sample Size and Efficiency of Sequential Ranking Procedures.* Theory of Probability and its Applications, 2012. (submitted)

3. I. N. Volodin. *Lower Bounds for Average Sample Size and Efficiency of Statistical Inference Procedures.* Theory of Probability and its Applications, 1979, vol. 24, No. 1, p. 120-129.

4. M. B. Malyutov. *Lower bounds for the mean length of a sequentially planned experiment.* Soviet Mathematics (Izvestiya VUZ. Matematika), 1983, vol. 27, No. 11, p. 21-47.

Asymptotic analysis in the large deviation zones for the distribution and density functions of the random sums

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Let's consider a weighted sum of a random number (r.n.) of summands

$$Z_N = \sum_{j=1}^{N} a_j X_j, \qquad Z_0 = 0,$$

where $\{X, X_j, j \geq 1\}$ is a family of i.i.d. random variables (r.vs.) with mean $\mathbf{E}X = \mu$, variance $\mathbf{D}X = \sigma^2$, distribution and density functions $F_X(x) = \mathbf{P}(X < x)$, $p_X(x) \leq C < \infty$, for all $x \in \mathbb{R}$, respectively. Here $C > 0$. In addition, it is assumed that $0 \leq a_j < \infty$, and a non-negative integer-valued random variable (r.v.) N with mean $\mathbf{E}N = \alpha$, variance $\mathbf{D}N = \beta^2$ and distribution $\mathbf{P}(N = l) = q_l$, $l \in \mathbb{N}_0$ is independent of X_j.
Since the appearance of the H.Robbins's results in 1948 the sums of the r.n. of summands have been investigated in the theory probability for quite some time. The principal results on asymptotic of the distributions of sums of the r.n. of r.vs. are summarized in [3].
Denote $T_{N,r} = \sum_{j=1}^{N} a_j^r$, $r \in \mathbb{N}$, $T_{0,r} = 0$, $r = 1, 2, \ldots$. It is clear, $\mathbf{E}T_{N,r} = \sum_{l=0}^{\infty} T_{l,r} q_l$, $\mathbf{D}T_{N,r} = \mathbf{E}T_{N,r}^2 - (\mathbf{E}T_{N,r})^2$. We say that X satisfies generalized Berntein's condition (B_γ), if there exist constants $\gamma \geq 0$ and $K > 0$ such that

$$|\mathbf{E}X^k| \leq (k!)^{1+\gamma} K^{k-2} \mathbf{E}X^2, \qquad k = 3, 4, \ldots. \qquad (B_\gamma)$$

Furthermore, we suppose that the r.vs. $T_{N,1}$, $T_{N,2}$ satisfy conditions (L) and (L_0), respectively, if there exist constants $K_1,\ K_2 > 0$ and $p \geq 0$ such that

$$|\Gamma_k(T_{N,1})| \leq (1/2) k! K_1^{k-2} (\mathbf{D}T_{N,1})^{1+(k-2)p}, \qquad k = 2, 3, \ldots, \qquad (L)$$

$$|\Gamma_k(T_{N,2})| \leq k! K_2^{k-1} (\mathbf{E}T_{N,2})^{1+(k-1)p}, \qquad k = 1, 2, \ldots. \qquad (L_0)$$

The first condition is used if $\mu \neq 0$, and the second one if $\mu = 0$.
We restrict our attention to the research of the upper estimate of the normal approximation to the sum $\tilde{Z}_N = (Z_N - \mathbf{E}Z_N)/(\mathbf{D}Z_N)^{1/2}$, large deviation theorems both in the Cramer and power Linnik zones and exponential inequalities for a tail probability $\mathbf{P}(\tilde{Z}_N \geq x)$. For the purpose the cumulant and characteristic function methods are used (see [4] for more details). Note that cumulant method was offered by V.Statulevičius in 1966.

Undoubtedly, there are a large amount of literature on theorems of large deviations for the random sums under different assumptions and with various applications (for example [1]), however in our knowledge, there are only a few papers, for example, V.Statulevičius (1973), L.Saulis and D.Deltuvienė (2007) on large deviations in the Cramer zone in case the cumulant method is used. Denote $a = \sup\{a_j, j = 1, 2, \ldots\} < \infty$, $(b \vee c) = \max\{b, c\}$, $b,\ c \in \mathbb{R}$. In the paper [2], we present the accurate upper estimate for the kth order cumulants and large deviation theorems for the distribution function of the sum $\tilde{Z}_N$ (see Lemma and Theorem below). In [2] only the case $\mu \neq 0$ was considered.

Lemma. *If for the r.v. X condition (B_γ) is fulfilled and the r.vs. $T_{N,1}$, $T_{N,2}$, satisfy conditions (L), (L_0), respectively, then*

$$|\Gamma_k(\tilde{Z}_N)| \leq (k!)^{1+\gamma}/\Delta_*^{k-2}, \qquad k = 3, 4, \ldots,$$

where

$$\Delta_* = \begin{cases} \Delta_N, & \text{if } \mu \neq 0, \\ \Delta_{N,0}, & \text{if } \mu = 0. \end{cases}$$

Here

$$\Delta_N = L_N^{-1}\sqrt{\mathbf{D}Z_N}, \quad L_N = 2\big(K_1|\mu|(\mathbf{D}T_{N,1})^p \vee (1 \vee \sigma/(2|\mu|))aM\big),$$

where $\mathbf{D}Z_N = \sigma^2\mathbf{E}T_{N,2} + \mu^2\mathbf{D}T_{N,1}$, $M = 2(\sigma \vee K)$.

$$\Delta_{N,0} = L_{N,0}^{-1}\sqrt{\mathbf{D}Z_N}, \quad L_{N,0} = 2aM\big(1 \vee K_2(\mathbf{E}T_{N,2})^p/(4a^2)\big),$$

where $\mathbf{D}Z_N = \sigma^2\mathbf{E}T_{N,2}$.

Theorem. *Let X and $T_{N,1}$, $T_{N,2}$ satisfy conditions (B_γ), (L), (L_0), respectively. Then relations*

$$\frac{1 - F_{\tilde{Z}_N}(x)}{1 - \Phi(x)} \to 1, \qquad \frac{F_{\tilde{Z}_N}(-x)}{\Phi(-x)} \to 1,$$

hold for $x \geq 0$, such that

$$x = \begin{cases} o((\mathbf{D}T_{N,1})^{(1/2-p)\nu(\gamma)}), & \text{if } \mu \neq 0, \\ o((\mathbf{E}T_{N,2})^{(1/2-p)\nu(\gamma)}), & \text{if } \mu = 0, \end{cases}$$

if $\mathbf{D}T_{N,1} \to \infty$, $\mathbf{E}T_{N,2} \to \infty$, *in case* $0 \le p < 1/2$. *Here* $\nu(\gamma) = (1+2(1\vee\gamma))^{-1}$, *and* $\Phi(x)$ *is the standard normal distribution function.*

References

1. O. Bonnin. Large deviation theorems for weighted compound Poisson sums. *Prob. and Math. Stat.*, 2003, vol. 23, p. 357–368.

2. A. Kasparavičiūtė, L. Saulis. Theorems on large deviations for randomly indexed sum of weighted random variables. *Acta Appl. Math.*, 2011, vol. 116, No. 3, p. 255–267.

3. V. M. Kruglov, V. Yu. Korolev. *Limit Theorems for Random Sums*, 1990, Moscow University: Moscow.

4. L. Saulis, V. Statulevičius. *Limit Theorems for Large Deviations*, 1991, Kluwer Academic Publishers: Dordrecht.

Tail conditional expectations for multivariate generalized Cox processes

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The main goal of our report is the estimation of impact of one portfolio component in full risk. This problem is very popular in financial mathematics (see for example [1] and [2]).

Let $N(t) = (N_1(t), \ldots, N_m(t))$ be a multivariate Poisson process (with dependent components in general), $\{X_j = (X_{j1}, \ldots, X_{jm})\}$ be a sequence of i.i.d. random vectors woth finite second moments, $\Lambda(t) = (\Lambda_1(t), \ldots, \Lambda_m(t))$ be a multivariate random process such that: $\Lambda_k(0) = 0$, $\Lambda_k(t)$ has nondecreasing paths, $E(\Lambda_k(t)) = b_k \cdot t$, $Var(\Lambda_k(t)) = s_k^2 \cdot t$, $b_k > 0$, $s_k^2 > 0$ for all $k = \overline{1, m}$. The processes $(N(t), t \ge 0)$ and $(\Lambda(t), t \ge 0)$ are independent.

We consider the following variant of multivariate generalized Cox process: $C(t) = (C_1(t), \ldots, C_m(t))$:

$$C_k(t) := \sum_{j=1}^{N_k(\Lambda_k(t))} X_{jk} \ .$$

Recently ([3]) we have proved the following

Theorem 1. *The convergence*

$$\frac{C(t) - A(t)}{\sqrt{t}} \Rightarrow Z \ , t \to \infty \ , \qquad (1)$$

holds for some random vector Z, where $A_k(t) = a_k \cdot l_k \cdot t$, if and only if the convergence

$$\frac{\Lambda(t) - \tilde{A}(t)}{\sqrt{t}} \Rightarrow V \ , t \to \infty \ , \tag{2}$$

holds for some random vector V, where $\tilde{A}_k(t) = l_k \cdot t$. Moreover

$$Z_k \stackrel{d}{=} \sqrt{l_k(\sigma_k^2 + a_k^2)} \cdot W_k + a_k \cdot V_k \ , \tag{3}$$

where $W = (W_1, \ldots, W_m)$ are i.i.d.r.v. with standard normal distributions and W and V are independent.

Our result is the analog of the result from [1].

Due to theorem 1 for large t we have

$$C_k(t) \stackrel{d}{=\approx} \sqrt{l_k(\sigma_k^2 + a_k^2)} \cdot Z \cdot \sqrt{t} + a_k \cdot \sqrt{t} \cdot V_k + a_k \cdot l_k \cdot t \ .$$

Let $S = C_1(t) + \ldots + C_m(t)$. It is easy to calculate

$$E(C_k(t)) = a_k \cdot l_k \cdot t + a_k \cdot \sqrt{t} \cdot V_k =: \mu_k \ , \ D(C_k(t)) = l_k(\sigma_k^2 + a_k^2) \cdot t =: d_k^2 \cdot t \ ,$$

$$\mu_S(t) = E(S(t)) = \sum_k \mu_k(t) \ , \ \sigma_S^2 = D(S(t)) = \sum_k d_k^2 \cdot t =: d^2 \cdot t \ .$$

Our main result is the following

Theorem 2. *Under above conditions for large t we have*

$$E(C_k(t)|S > x) \approx a_k l_k t + E_V[\lambda(A(x,t) - \sum_k a_k V_k / d)] \frac{d_k^2}{d} \sqrt{t} \ , \tag{4}$$

where

$$A(x,t) = \frac{x - \mu_S t}{d\sqrt{t}} \ .$$

In the case of independent increments of the process $\Lambda(t)$ random vector V has multivariate normal distribution and we can calculate expression (4) explicitely using the result from [2].

References

1. Landsman Z., Valdes E. A., Tail Conditional Expectations for Elliptical Distributions. *University of Haifa, Technical Report 02-04*, October 2002.

2. Panjer H. H., Measurement of Risk, Solvency Requirements and Allocation of Capital within Financial Conglomerates. *Institute of Insurance and Pension Research, University of Waterloo, Research Report 01-15*, 2002.

3. Yury Khokhlov, Olga Rumyantseva. Multivariate generalized Cox process. *In: XXIX International Seminar on Stability Problems for Stochastic Models and V International Workshop "Applied Problems in Theory of Probabilities and Mathematical Statistics related to modeling of Information Systems"*, 10-16 October 2011, Svetlogorsk, Russia. – Book of Abstracts. – p. 29.

4. Bening V.E., Korolev V.Yu. *Generalized Poissin Models and Their Applications in Insurance and Finance*, 2002, Utrecht, VSP.

Multivariate analog of Birnbaum-Saunders distribution

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The univariate family of distributions proposed by Birnbaum and Saunders ([1]), also known as the fatigue life distributions, has been widely applied for describing fatigue lifetimes. This family was originally derived from a model for which failure follows from the development and growth of a dominant crack (see also [2]). But they consider the length of a crack only but not its direction. We propose a multivariate analog of this model where we consider the development of a crack in space. A few papers are devoted to multivariate versions for BS distribution and all of them have used the analytical approach (see [3], [4]). We follow original approach from [1].

Let $\{\xi_n\}$ be a sequence of independent identically distributed random vectors in R^m with mean vector $\mu = (\mu_1, \ldots, \mu_m)^T$ and covariance matrix $A = (a_{ij})$, $g(x), x \in R^m$ be continious real-valued positive function. Following paper [2] we consider the process of development of crack in R^m:

$$X_{k+1} = X_k + \xi_{k+1} \cdot g(X_k) \ , \ X_0 = 0.$$

Fix some vector $h = (h_1, \ldots, h_m)^T$ with positive components and consider random vector $\tau = (\tau_1, \ldots, \tau_m)^T$, where τ_j is the passage time of the level h_j by $X_{k,j}$. Next let $t = (t_1, \ldots, t_m) = (u \cdot s_1, \ldots, u \cdot s_m)$ be vector with positive components, where (without loss of generality) we assume $t_1 < \ldots < t_m$. Define vector $a(h) = (a_1(h_1), \ldots, a_m(h_m))^T$ with components

$$a_j(h_j) = \int_0^{h_j} \frac{dx_j}{g(x} \ ,$$

vector $\mu \circ t := (\mu_1 \cdot t_1, \ldots, \mu_m \cdot t_m)^T$ and matrix $A \circ t$ by the rule: all elememnts of first column and row are multiplied by t_1, next all elements of second column and row (with the exception of used ones) are multiplied by t_2 and so on.

Using the method from the paper [2] it can be shown that for large u

$$P(\tau > t) \approx \Phi((A \circ t)^{-1/2}(a(h) - \mu \circ t)) \ , \qquad (1)$$

where $\Phi(x), x \in R^m$ is the distribution function of standard normal distribution in R^m.

The expression (2) defines of multivariate analog Birbaum-Saunders distribution.

In our report we give more explicite description of this distribution and investigate its properties.

References

1. Birnbaum Z. W., Saunders S. C. A new family of life distribution. *J.Appl.Probab.*, 1962, vol. 6, p. 319-327.

2. Desmond A. Stochastic models of failure in random environments. *Canadian J. of Statist.*, 1985, vol. 13, p. 171-183.

3. Debasis Kundu, N. Balakrishnan, A. Jamalizadeh. Bivariate Birnbaum-Saunders distribution and associated inference. *Journal of Multivariate Analysis*, 2010, vol. 101, p. 113-125.

4. Artur J. Lemonte, Guillernmo Martinez-Florez, German Moreno-Arenas. Skewed Multivariate Birnbaum-Saunders Distributions. 2012 (preprint, arXiv: 1204.628v1 [stat.ME]).

A limiting description of parallel minimax control in a random environment (two-armed bandit problem)

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Let's consider a computer system processing a large number T items of data. Two universal methods of data processing are available, numbered by $\ell = 1, 2$. One can consider a result of processing of the t-th item of data as a current value of a random controlled process ξ_t, $t = 1, \ldots, T$, which values depend only on currently chosen method ℓ and are often interpreted as current incomes. The goal of the control is to maximize or to minimize (in some sense) the total expected income. Values of the process may have different meanings. For example, they may be equal to the duration of the processing. Or they may have binary values: $\xi_t = 1$ if the processing is successful and $\xi_t = 0$ if it is not. In the first case the goal is to minimize the total expected duration of the processing and in the second case the goal is to maximize the total expected successfully processed data.

The core of the problem is that the best method is not known in advance because it may be different for different data. So, it should be estimated meanwhile the control process. This is the problem of rational adaptive control in a random environment which is also well-known as the two-armed bandit problem (see e.g. Sragovich [1], Berry and Fristedt [2]). The usual approach to the control is to process data sequentially, one by one. However, if the problem is considered in minimax setting it turned out that the control may be implemented in parallel almost without the lack of its quality, i.e. under mild conditions minimax risks in both cases of parallel and sequential controls have close values. For example, $T = 10^6$ items of data may be partitioned into $N = 50$ groups each containing $K = 2 \cdot 10^4$ items of data so that data in each group are processed in parallel and the results of processing are summarized. Calculations show that $N = 50$ or even $N = 30$ provides a high quality of the control. A direct determination of minimax strategy and minimax risk is practically impossible. However, it is shown in Kolnogorov [3, 4] that they can be found as Bayes' ones corresponding to the worst prior distribution on the set of parameters. The strategy can be determined numerically and has a simple threshold type. The results are explicit ones if ξ_t, $t = 1, \ldots, T$ are normally distributed. However, according to the central limit theorem the summarized incomes of groups of data may have distributions close to normal even if original distributions of ξ_t, $t = 1, \ldots, T$ were not those.

A sequential design of optimal minimax control and its limiting description are considered. The results of numerical experiments and Monte Carlo simulations are given.

References

1. V. G. Sragovich. *Adaptive Control*, 1981, Nauka, Moscow. (In Russian)

2. D. A. Berry, B. Fristedt. *Bandit problems*, 1985, Chapman and Hall, London, New York.

3. A. V. Kolnogorov. Finding Minimax Strategy and Minimax Risk in a Random Environment (the Two-Armed Bandit Problem). *Automation and Remote Control*, 2011, vol. 72, No. 5, p. 1017-1027.

4. A. V. Kolnogorov. Parallel Design of Robust Control in the Stochastic Environment (the Two-Armed Bandit Problem). *Automation and Remote Control*, 2012, vol. 73, No. 4, p. 689-701.

Nonlinear Markov processes and mean field games

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Highlights:

(i) Nonlinear Markov semigroup is a nonlinear deterministic dynamic system on the set of measures preserving positivity.

(ii) Nonlinear Markov process can be defined as (a) family of processes (parametrized by initial distributions) s.t. to each trajectory there corresponds a 'tangent' (time non-homogeneous) Markov process; (b) future depends on the past via its present position and distribution.

(iii) The first derivative with respect to initial data describes the interacting particle approximation (to which the nonlinear dynamics serves as the dynamic LLN).

(iv) The second derivative describes the limit of fluctuations of the evolution of particle systems around its LLN (probabilistically the dynamic CLT).

(v) Controlled version lead to nonlinear Markov (or measure-valued) control arising in the limit of large number controlled interacting particle systems.

Plan of the talk:

(i) Nonlinear Markov semigroups and processes: definitions and examples.

(ii) Well posedness and sensitivity analysis.

(iii) Interacting particles: mean field and k-ary interactions. Nonlinear Markov processes as dynamic LLN.

(v) Fluctuations and CLT

(vi) Further developments: nonlinear Markov control and mean field games.

References

1. V. N. Kolokoltsov. *Nonlinear Markov processes and kinetic equations*. Cambridge University Press, August 2010.

2. V. N. Kolokoltsov. *Markov processes, semigroups and generators*. DeGruyter, 2011.

3. V. N. Kolokoltsov. The central limit theorem for the Smoluchovski coagulation model. Prob. Theory Relat. Fields 146: 1 (2010), 87-153.

4. V.N. Kolokoltsov. The Lévy-Khintchine type operators with variable Lipschitz continuous coefficients generate linear or nonlinear Markov processes and semigroups. Prob. Theory Relat. Fields. Online First 2011.

5. V. N. Kolokoltsov. Nonlinear Markov games. Proceedings of the 19th MTNS Symposium, Budapest July 2010.

6. V. N. Kolokoltsov. Nonlinear Lévy and nonlinear Feller processes: an analytic introduction, 2011. http://arxiv.org/abs/1103.5591

7. V. N. Kolokoltsov. Nonlinear Markov games on a finite state space (mean-field and binary interactions) http://arxiv.org/abs/1105.3053. International Journal of Statistics and Probability. Canadian Center of Science and Education (Open access journal), **1:1** (2012), 77-91. http://www.ccsenet.org/journal/index.php/ijsp/article/view/16682

8. Vassili N. Kolokoltsov, Jiajie Li and Wei Yang. Mean Field Games and Nonlinear Markov Processes (2011). arXiv:1112.3744

On convergence of the distributions of random sums and statistics constructed from samples with random sizes to exponential power laws

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Let $0 < \alpha \leqslant 2$. *Exponential power distribution* is the absolutely continuous distribution defined by its Lebesgue probability density

$$\ell_\alpha(x) = \frac{\alpha}{2\Gamma(\frac{1}{\alpha})} \cdot e^{-|x|^\alpha}, \quad -\infty < x < \infty. \tag{1}$$

With $\alpha = 1$ relation (1) defines the classical Laplace distribution with zero mean and variance 2. With $\alpha = 2$ relation (1) defines the normal (Gaussian) distribution with zero mean and variance $\frac{1}{2}$.

The class of distributions (1) was introduced and studied by M. T. Subbotin in 1923 [1]. Along with the term *generalized Laplace distribution* going back to the original paper [1] at least three other different terms are used for distribution (1). For example, in [2] this distribution is called *exponential power distribution*, in [3] and [4] it is called *generalized error distribution* whereas in [5] the term *generalized exponential distribution* is used. Distributions of type (1) are widely used in Bayesian analysis and various applications from astronomy to signal and image processing.

Probably, by now the simplicity of representation (1) has been the main (at least, important) reason for using the exponential power distributions in many applied problems as a heavy-tailed (for $0 < \alpha < 2$) alternative to the normal law. The "asymptotic" reasons of possible adequacy of this model have not been provided yet. Here we demonstrate that the exponential power distribution can be limiting in rather simple limit theorems for regular statistics constructed from samples with random sizes, in particular, in the scheme of random summation. Hence, along with the normal law, this distribution can be regarded as an asymptotic approximation for the distributions of some processes, say, similar to (non-homogeneous) random walks.

By $g_{\alpha,\theta}(x)$ we denote the probability density of the strictly stable law with characteristic exponent α and parameter θ defined by the characteristic function

$$\mathfrak{f}_{\alpha,\theta}(t) = \exp\Big\{-|t|^{\alpha}\exp\Big\{-\frac{i\pi\theta\alpha}{2}\mathrm{sign}t\Big\}\Big\}, \qquad t \in \mathbb{R}, \tag{2}$$

with $0 < \alpha \leqslant 2$, $|\theta| \leqslant \theta_{\alpha} = \min\{1, \frac{2}{\alpha} - 1\}$ (see, e. g., [6]. The standard normal distribution function will be denoted $\Phi(x)$. Denote

$$h_{\alpha/2}(z) = \frac{\alpha}{\Gamma(\frac{1}{\alpha})}\sqrt{\frac{\pi}{2}}\cdot\frac{g_{\alpha/2,1}(z)}{\sqrt{z}}, \quad z > 0,$$

$$w_{\alpha/2}(z) = z^{-2}h_{\alpha/2}(z^{-1}) = \frac{\alpha}{\Gamma(\frac{1}{\alpha})}\sqrt{\frac{\pi}{2}}\cdot\frac{g_{\alpha/2,1}(z^{-1})}{z^{3/2}}, \quad z > 0.$$

It can be easily verified that $h_{\alpha/2}(z)$ and $w_{\alpha/2}(z)$ are the probability densities of nonnegative random variables. The distribution functions corresponding to the densities $\ell_{\alpha}(x)$, $h_{\alpha/2}(z)$ and $w_{\alpha/2}(z)$ will be denoted by the corresponding capital letters: $L_{\alpha}(x)$, $H_{\alpha/2}(z)$ and $W_{\alpha/2}(z)$. The symbol $\stackrel{d}{=}$ will stand for the coincidence of distributions.

Lemma 1. *Exponential power distribution* (1) *is a scale mixture of normal laws*:

$$L_{\alpha}(x) = \int_0^{\infty}\Phi\big(x\sqrt{z}\big)dH_{\alpha/2}(z), \tag{3}$$

$$L_{\alpha}(x) = \int_0^{\infty}\Phi\Big(\frac{x}{\sqrt{z}}\Big)dW_{\alpha/2}(z). \tag{4}$$

If Z_{α} is a random variable having the exponential power distribution with parameter α, then $Z_{\alpha} \stackrel{d}{=} X\cdot\sqrt{U_{\alpha/2}}$, where X and $U_{\alpha/2}$ are independent random variables such that X has the standard normal distribution, $U_{\alpha/2} \stackrel{d}{=} V_{\alpha/2}^{-1}$, and $V_{\alpha/2}$ is an absolutely continuous random variable whose probability density is $h_{\alpha/2}(z)$.

Consider a sequence of independent identically distributed random variables $X_1, X_2, \ldots$, defined on a probability space $(\Omega, \mathfrak{A}, \mathsf{P})$. Assume that $\mathsf{E}X_1 = 0$, $0 < \sigma^2 = \mathsf{D}X_1 < \infty$. For a natural $n \geqslant 1$ let $S_n = X_1 + \ldots + X_n$. Let

$N_1, N_2, \ldots$ be a sequence of nonnegative integer random variables defined on the same probability space so that for each $n \geqslant 1$ the random variable N_n is independent of the sequence $X_1, X_2, \ldots$ A random sequence $N_1, N_2, \ldots$ is said to be infinitely increasing ($N_n \longrightarrow \infty$) in probability, if $\mathsf{P}(N_n \leqslant m) \longrightarrow 0$ as $n \to \infty$ for any $m \in (0, \infty)$.

Lemma 2. *Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above and $N_n \longrightarrow \infty$ in probability as $n \to \infty$. A distribution function $F(x)$ such that*

$$\mathsf{P}\left(\frac{S_{N_n}}{\sigma\sqrt{n}} < x\right) \Longrightarrow F(x) \quad (n \to \infty),$$

exists if and only if there exists a distribution function $Q(x)$ satisfying the conditions $Q(0) = 0$,

$$F(x) = \int\limits_0^\infty \Phi\left(\frac{x}{\sqrt{y}}\right) dQ(y), \quad x \in \mathbb{R}, \quad \mathsf{P}(N_n < nx) \Longrightarrow Q(x) \ \ (n \to \infty).$$

Proof. This lemma was proved in [8]

Theorem 1. *Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above and $N_n \longrightarrow \infty$ in probability as $n \to \infty$. Then*

$$\mathsf{P}\left(\frac{S_{N_n}}{\sigma\sqrt{n}} < x\right) \Longrightarrow L_\alpha(x) \quad (n \to \infty),$$

if and only if

$$\mathsf{P}(N_n < nx) \Longrightarrow W_{\alpha/2}(x) \ \ (n \to \infty).$$

This statement is a direct consequence of lemma 2 with $Q(x) = W_{\alpha/2}(x)$ and representation (4).

For $n \geqslant 1$ let $T_n = T_n(X_1, \ldots, X_n)$ be a statistic, that is, a measurable function of the random variables $X_1, \ldots, X_n$. For each $n \geqslant 1$ define the random variable T_{N_n} by letting $T_{N_n}(\omega) = T_{N_n(\omega)}\big(X_1(\omega), \ldots, X_{N_n(\omega)}(\omega)\big)$ for every elementary outcome $\omega \in \Omega$. We will say that the statistic T_n is asymptotically normal, if there exist $\delta > 0$ and $\theta \in \mathbb{R}$ such that

$$\mathsf{P}\left(\delta\sqrt{n}\big(T_n - \theta\big) < x\right) \Longrightarrow \Phi(x) \quad (n \to \infty). \tag{5}$$

Lemma 3. *Assume that $N_n \longrightarrow \infty$ in probability as $n \to \infty$. Let the statistic T_n be asymptotically normal in the sense of* (5)*. Then a distribution function $F(x)$ such that*

$$\mathsf{P}\left(\delta\sqrt{n}\big(T_{N_n} - \theta\big) < x\right) \Longrightarrow F(x) \quad (n \to \infty),$$

exists if and only if there exists a distribution function $Q(x)$ satisfying the conditions $Q(0) = 0$,

$$F(x) = \int_0^\infty \Phi\big(x\sqrt{y}\big)dQ(y), \quad x \in \mathbb{R}, \quad \mathsf{P}(N_n < nx) \Longrightarrow Q(x) \quad (n \to \infty).$$

This lemma is a particular case of theorem 3 in [9], the proof of which is, in turn, based on general theorems on convergence of superpositions of independent random sequences [10]. Also see [7], theorem 3.3.2.

Theorem 2. *Assume that $N_n \longrightarrow \infty$ in probability as $n \to \infty$. Let the statistic T_n be asymptotically normal in the sense of* (5)*. Then*

$$\mathsf{P}\left(\delta\sqrt{n}\big(T_{N_n} - \theta\big) < x\right) \Longrightarrow L_\alpha(x) \quad (n \to \infty),$$

if and only if

$$\mathsf{P}(N_n < nx) \Longrightarrow H_{\alpha/2}(x) \quad (n \to \infty).$$

This statement is a direct consequence of lemma 3 with $Q(x) = H_{\alpha/2}(x)$ and representation (3).

We also give simple examples of mixed Poisson random variables satisfying the conditions of theorems 1 and 2.

This research was supported by the Russian Foundation for Basic Research (projects 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a), and by the Ministry for Education and Science of Russian Federation (state contract 16.740.11.0133).

References

1. *Subbotin M. T.* On the law of frequency of error // Matematicheskii Sbornik, 1923. Vol. 31. No. 2. P. 296–301.

2. *Box G., Tiao G.* Bayesian Inference in Statistical Analysis. – Reading: Addison–Wesley, 1973.

3. *Evans M., Hastings N., Peacock B.* Statistical Distributions, 3rd. Ed. – New York: Wiley, 2000.

4. *Leemis L. M., McQueston J. T.* Univariate distribution relationships // The American Statistician, 2008. Vol. 62. No. 1. P. 45–53.

5. RiskMetrics Technical Document. – New York: RiskMetric Group, J.P. Morgan, 1996.

6. *Zolotarev V. M.* One-dimensional Stable Distributions. – Providence, RI: American Mathematical Society, 1986.

7. *Gnedenko B. V., Korolev V. Yu.* Random Summation: Limit Theorems and Applications // Boca Raton: CRC Press, 1996.

8. *Korolev V. Yu.* Convergence of random sequences with independent random indices. I // Theory Probab. Appl., 1994. Vol. 39, No. 2. P. 313–333.

9. *Korolev V. Yu.* Convergence of random sequences with independent random indices. II // Theory Probab. Appl., 1995. Vol. 40, No. 4. P. 907–910.

10. *Korolev V. Yu.* A general theorem on the limit behavior of superpositions of independent random processes with applications to Cox processes // Journal of Mathematical Sciences, 1996. Vol. 81, No. 5. P. 2951–2956.

On convergence of random walks having jumps with finite variances to stable Lévy processes

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In [1, 2] some functional limit theorems were proved for compound Cox processes with square integrable leading random measures. However, the class of limit processes for compound Cox processes having jumps with finite variances and such leading random measures cannot contain any stable Lévy process besides the Wiener process. The aim of the present work is to fill this gap.

Let $D = D[0,1]$ be a space of real-valued right-continuous functions defined on $[0,1]$ and having left-side limits. Let $\mathcal{F}$ be the class of strictly increasing continuous mappings of the interval $[0,1]$ onto itself. Let f be a non-decreasing function on $[0,1]$, $f(0)=0$, $f(1)=1$. Let

$$\|f\| = \sup_{s\neq t}\left|\log\frac{f(t)-f(s)}{t-s}\right|.$$

If $\|f\|<\infty$, then the function f is continuous and strictly increasing, hence, it belongs to $\mathcal{F}$.

Define the metric $d_0(x,y)$ in $D[0,1]$ as the greatest upper bound of positive numbers ϵ for which $\mathcal{F}$ contains a function f such that $\|f\|\leqslant\epsilon$ and

$$\sup_t |x(t)-y(f(t))| \leqslant \epsilon.$$

It can be shown that $D[0,1]$ is complete with respect to the metric d_0. The metric space $(D[0,1], d_0)$ is referred to as the *Skorokhod space.*

We will consider stochastic processes as random elements in $\mathcal{D} \equiv (D[0,1],\, d_0)$ in the following sense. Let $\mathfrak{D}$ be the class of Borel sets of the space $\mathcal{D}$. The class $\mathfrak{D}$ is the σ-algebra generated by the open sets of $\mathcal{D}$. A mapping X of the basic probability space $(\Omega,\, \mathcal{A},\, \mathsf{P})$ to $\mathcal{D}$ is measurable if $\{\omega:\, X(\omega) \in B\} \in \mathcal{A}$ for any set $B \in \mathfrak{D}$. By a stochastic process we will mean a measurable mapping X of Ω to $\mathcal{D}$. By the distribution of a stochastic process we will mean the probability measure P^X on the measurable space $(\mathcal{D},\, \mathfrak{D})$ defined for any set $A \in \mathfrak{D}$ by the relation $\mathsf{P}^X(A) = \mathsf{P}\big(\{\omega:\, X(\omega) \in A\}\big) \equiv \mathsf{P}(X \in A)$. The symbol $\Longrightarrow$ will denote weak convergence: the sequence $\{X_n(t)\}_{n\geqslant 1}$ of stochastic processes weakly converges to a stochastic process $X(t)$, that is, $X_n(t) \Longrightarrow X(t)$, if

$$\int w(\omega)\mathsf{P}^{X_n}(d\omega) \longrightarrow \int w(\omega)\mathsf{P}^{X}(d\omega)$$

for any continuous bounded function w.

By a Lévy process, as usual, we will mean a homogeneous stochastically continuous stochastic process $X(t)$, $t \in [0,1]$, with independent increments such that $X(0) = 0$ a.s. and the sample paths $X(t) \in D[0,1]$. As is easily seen, for each $t \in [0,1]$ the random variable $X(t)$ has an infinitely divisible distribution.

The strictly stable distribution function with the characteristic exponent $\alpha \in (0,2]$ and parameter θ ($|\theta| \leqslant \theta_\alpha = \min\{1, \frac{2}{\alpha} - 1\}$) determined by the characteristic function

$$\mathfrak{g}_{\alpha,\theta}(s) = \exp\Big\{-|s|^\alpha \exp\Big\{-\frac{i\pi\theta\alpha}{2}\mathrm{sign} s\Big\}\Big\}, \qquad s \in \mathbb{R},$$

will be denoted $G_{\alpha,\theta}(x)$. The value $\theta = 0$ corresponds to symmetric strictly stable laws. The values $\theta = 1$ and $0 < \alpha \leqslant 1$ correspond to one-sided strictly stable distributions. As is known, if ξ is a random variable with the distribution function $G_{\alpha,\theta}(x)$, $0 < \alpha < 2$, then $\mathsf{E}|\xi|^\delta < \infty$ for any $\delta \in (0,\alpha)$, but the moments of orders greater or equal to α of the random variable ξ do not exist (see, e.g., [3]).

The distribution function of the standard normal law ($\alpha = 2$, $\theta = 0$) will be denoted $\Phi(x)$. It is well known that

$$G_{\alpha,0}(x) = \int_0^\infty \Phi\Big(\frac{x}{\sqrt{u}}\Big) dG_{\alpha/2,1}(u), \quad x \in \mathbb{R} \tag{1}$$

(see, e.g., [3] or [4]). To representation (1) there corresponds the representation in terms of characteristic functions:

$$\mathfrak{g}_{\alpha,0}(s) = \int_0^\infty \exp\Big\{-\frac{s^2 u}{2}\Big\} dG_{\alpha/2,1}(u), \quad s \in \mathbb{R}. \tag{2}$$

A Lévy process $X(t), t \in [0,1]$, is called *α-stable*, if $\mathsf{P}\big(X(1) < x\big) = G_{\alpha,\theta}(x)$, $x \in \mathbb{R}$. It can be shown that if $X(t)$, $t \in [0,1]$, is a Lévy process, then $X(t)$ is α-stable if and only if $X(t) \stackrel{d}{=} t^{1/\alpha}X(1)$, $t \in [0,1]$ (see, e.g., [5]).

Consider a sequence of compound Cox processes

$$Z_n(t) = \sum_{i=1}^{N_1^{(n)}(\Lambda_n(t))} X_{n,i}, \quad t \geqslant 0, \tag{3}$$

where $\{N_1^{(n)}(t),\ t \in [0,1]\}_{n\geqslant 1}$ are Poisson processes with unit intensity; for each $n = 1,2,...$ the random variables $X_{n,1}, X_{n,2},...$ are identically distributed, moreover, for each $n \geqslant 1$ the random variables $X_{n,1}, X_{n,2},...$ and the process $N_1^{(n)}(t)$, $t \in [0,1]$, are independent; for each $n = 1,2,...$ the random measure $\Lambda_n(t)$, $t \in [0,1]$, is a Lévy process independent of the process

$$X_n(t) = \sum_{i=1}^{N_1^{(n)}(t)} X_{n,i}, \quad t \geqslant 0,$$

such that $\Lambda_n(0) = 0$, $\Lambda_n(1) \stackrel{d}{=} k_n U_{\alpha,1}^{(n)}$, where $\{k_n\}_{n\geqslant 1}$ is an infinitely increasing sequence of natural numbers and $U_{\alpha,1}^{(1)}, U_{\alpha,1}^{(2)}, ...$ is a sequence of identically distributed a.s. positive random variables having one-sided strictly stable distribution with parameters $\alpha \in (0,1]$ and $\theta = 1$. For definiteness, we assume that $\sum_{i=1}^{0} = 0$. From the abovesaid it follows that $\mathsf{E}\Lambda_n^{\beta}(1) < \infty$ for any $\beta < \alpha$ and

$$\Lambda_n(t) \stackrel{d}{=} t^{1/\alpha}\Lambda_n(1) \stackrel{d}{=} t^{1/\alpha}k_n U_{\alpha,1}^{(n)} \stackrel{d}{=} t^{1/\alpha}k_n U_{\alpha,1}^{(1)}, \quad t \geqslant 0. \tag{4}$$

Assume that

$$\mathsf{E}X_{n,1} = 0 \ \text{ and } \ 0 < \sigma_n^2 \equiv \mathsf{E}X_{n,1}^2 < \infty. \tag{5}$$

Let $t = 1$. Denote $N_n = N_1^{(n)}(\Lambda_n(1))$. Assume that, as $n \to \infty$,

$$\mathsf{P}(X_{n,1} + ... + X_{n,k_n} < x) \longrightarrow \Phi(x), \tag{6}$$

with the same $\{k_n\}_{n\geqslant 1}$ as in the definition of the random measures $\Lambda_n(t)$. From the classical theory of limit theorems it is known that (6) holds, if, as $n \to \infty$,

$$k_n\sigma_n^2 \longrightarrow 1 \tag{7}$$

and

$$k_n\mathsf{E}X_{n,1}^2\mathbb{I}(|X_{n,1}| \geqslant \epsilon) \longrightarrow 0 \tag{8}$$

for any $\epsilon > 0$.

Moreover, by virtue of (4) it is obvious that

$$\frac{\Lambda_n(1)}{k_n} \stackrel{d}{=} \frac{k_n U_{\alpha,1}^{(1)}}{k_n} = U_{\alpha,1}^{(1)}.$$

therefore, formally,

$$\frac{\Lambda_n(1)}{k_n} \Longrightarrow U_{\alpha,1}^{(1)}. \tag{9}$$

But, as it was shown in [6] (also see, e.g., [7] or [8]), (9) is equivalent to

$$\frac{N_n}{k_n} \Longrightarrow U_{\alpha,1}^{(1)}. \tag{10}$$

By the Gnedenko–Fahim transfer theorem [9] (also see theorem 2.9.1 in [8]) conditions (6) and (10) imply that, as $n \to \infty$,

$$Z_n(1) = X_{n,1} + \ldots + X_{n,N_n} \Longrightarrow Z, \tag{11}$$

where Z is the random variable with the characteristic function

$$\mathfrak{f}(s) = \int\limits_0^\infty \exp\Big\{-\frac{s^2 u}{2}\Big\} d\mathsf{P}(U_{\alpha,1}^{(1)} < u), \quad s \in \mathbb{R}.$$

But by virtue of (2)

$$\mathfrak{f}(s) = \int\limits_0^\infty \exp\Big\{-\frac{s^2 u}{2}\Big\} dG_{\alpha,1}(u) = \mathfrak{g}_{2\alpha,0}(s), \quad s \in \mathbb{R},$$

that is, the limit random variable Z in (11) has the symmetric strictly stable distribution with the characteristic exponent $\alpha_0 = 2\alpha$.

Consider an α_0-stable Lévy process $Z(t)$, $t \in [0,1]$, such that $Z(1) \stackrel{d}{=} Z$. Since $Z_n(t)$ and $Z(t)$ are Lévy processes, almost all their sample paths belong to the Skorokhod space $\mathcal{D}$.

Using theorem 15.6 from [10] we obtain the following result.

THEOREM. *Let $\alpha \in (0,1]$ and a compound Cox process $Z_n(t)$ (see (3)) be controlled by the Lévy process $\Lambda_n(t)$ such that $\Lambda_n(1) \stackrel{d}{=} k_n U_{\alpha,1}^{(n)}$, where $\{k_n\}_{n\geqslant 1}$ is an infinitely increasing sequence of natural numbers and $U_{\alpha,1}^{(1)}, U_{\alpha,1}^{(2)}, \ldots$ is a sequence of identically distributed a.s. positive random variables having one-sided strictly stable distribution with parameters α and $\theta = 1$. Assume that the random jumps $\{X_{n,j}\}_{j\geqslant 1}$, $n = 1, 2, \ldots$, of the compound Cox process $Z_n(t)$ satisfy conditions (5), (7) and (8) with the same numbers k_n. Then the random walks generated by these compound Cox processes weakly converge in the Skorokhod space $\mathcal{D} = (D[0,1], d_0)$ to a 2α-stable Lévy process $Z(t)$ with $\mathsf{P}\big(Z(1) < x\big) = G_{2\alpha,0}(x)$.*

This research was supported by the Russian Foundation for Basic Research (projects 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a), and by the Ministry for Education and Science of Russian Federation (state contract 16.740.11.0133).

References

1. *Kashcheev D. E.* Functional limit theorems for compound Cox processes // Surveys in Applied and Industrial mathematics, 2000. Vol. 7. No. 2, P. 494–495 (in Russian).

2. *Kashcheev D. E.* Modelling the Dynamics of Financial Time Series and Pricing Financial Derivatives. PhD. Thesis. – Tver: Tver State University, 2001 (in Russian).

3. *Zolotarev V. M.* One-Dimensional Stable Distributions. – Providence: American Mathematical Society, 1986.

4. *Feller W.* An Introduction to Probability Theory and its Applications. Vol. 2. – New York: Wiley, 1971.

5. *Embrechts P., Maejima M.* Selfsimilar Processes. – Princeton: Princeton University Press, 2002.

6. *Gnedenko B. V., Korolev V. Yu.* Random Summation: Limit Theorems and Applications. – Boca Raton: CRC Press, 1996.

7. *Bening V., Korolev V.* Generalized Poisson Models and their Applications in Insurance and Finance. – Utrecht: VSP, 2002.

8. *Korolev V. Yu., Bening V. E., Shorgin S. Ya.* Mathematical Foundations of Risk Theory. 2nd Ed. – Moscow: FIZMATLIT, 2011 (in Russian).

9. *Gnedenko B. V., Fahim H.* On a transfer theorem // Soviet Math. Dokl., 1969. Vol. 187. No. 1. P. 15–17.

10. *Billingsley P.* Convergence of Probability Measures. – New York: Wiley, 1968.

On convergence of the distributions of random sums to generalized variance gamma distributions

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In this communication we demonstrate that any distribution belonging to the class of generalized variance gamma distributions, the class of variance-mean mixtures of normal laws with generalized gamma mixing distributions, can be limiting in rather simple limit theorems for random sums of i.i.d. random variables (r.v.'s). This class is more general and flexible than the class of generalized hyperbolic distributions widely used in various applied problems.

The class of generalized gamma-distributions (GG-distributions) was first described in 1962 in [1] as a family which contains both gamma-distributions and Weibull distributions. This class is similar to the class of generalized inverse Gaussian distributions but, unlike the latter, it contains distributions whose densities have tails with exponential-power type of decrease.

The GG-distribution is defined by its density

$$f(x;\nu,\kappa,\delta)=\frac{|\nu|}{\delta\Gamma(\kappa)}\Big(\frac{x}{\delta}\Big)^{\kappa\nu-1}\exp\Big\{-\Big(\frac{x}{\delta}\Big)^{\nu}\Big\},\quad x\geqslant 0, \tag{1}$$

with parameters $\nu\in\mathbb{R}$, $\kappa>0$ and $\delta>0$ responsible for *exponential power, shape* and *scale*, where $\Gamma(\kappa)=\int_0^\infty x^{\kappa-1}e^{-x}dx$ is Euler's gamma-function.

The class of GG-distributions includes practically all most widely used absolutely continuous distributions concentrated on $\mathbb{R}_+$. In particular, it contains gamma-distributions ($\nu=1$) including exponential ($\nu=1$, $\kappa=1$), Erlang ($\nu=1$, $\kappa\in\mathbb{N}$) and chi-square ($\nu=1$, $\delta=2$) distributions, Nakagami distributions ($\nu=2$), half-normal distribution (the distribution of the absolute value of a standard normal r.v. or, which is the same, the distribution of the maximum of the standard Wiener process on $[0,1]$) ($\nu=2$, $\kappa=\frac{1}{2}$), Rayleigh distributions ($\nu=2$, $\kappa=1$), chi-distributions ($\nu=2$, $\delta=\sqrt{2}$), Maxwell distribution ($\nu=2$, $\kappa=3/2$), Weibull distribution ($\kappa=1$), inverse gamma-distributions ($\nu=-1$) including Lévy distribution ($\nu=-1$, $\kappa=1/2$). The lognormal distribution is the limiting case of GG-distribution with $\kappa\to\infty$. The wide applicability of GG-distributions can be explained by that almost all of them can be limiting in various limit theorems of probability theory, e.g., dealing with the schemes of summation or maximum of independent r.v.'s.

In [2], the family of distributions

$$W(x;\alpha,\nu,\kappa,\delta)=\int\limits_0^\infty\Phi\bigg(\frac{x-\alpha u}{\sqrt{u}}\bigg)f(u;\nu,\kappa,\delta)du, \tag{2}$$

was introduced, where $f(u;\nu,\kappa,\delta)$ is the density of GG-distribution (see (1)). In [2] distributions (2) were called *generalized variance gamma distributions* (GVG distributions). The class of GVG distributions contains many generalized hyperbolic distributions including variance gamma distributions (with mixing gamma-distributions), skew Student distributions (with mixing inverse gamma-distributions), normal\\inverse Gaussian distributions (with mixing Lévy distributions). But along with these laws, GVG distributions also include variance-mean mixtures of normal laws with Weibull-type mixing distributions in which the exponential power can be arbitrary.

Assume that all the r.v.'s discussed below are defined on the same probability space $(\Omega,\,\mathfrak{A},\,\mathsf{P})$. Let $\{X_{n,j}\}_{j\geqslant1}$, $n=1,2,...$ be a double array of row-wise i.i.d. r.v.'s. Let $\{N_n\}_{n\geqslant 1}$ be a sequence of nonnegative integer-valued r.v.'s independent of $X_{n,1}, X_{n,2}, \ldots$ for each $n \geqslant 1$. Let $S_{n,k} = X_{n,1} + \ldots + X_{n,k}$. To avoid misunderstanding, assume $\sum_{j=1}^{0} = 0$. The symbol $\Longrightarrow$ will denote convergence in distribution.

The Lévy distance which metrizes convergence in distribution in the space of distribution functions (d.f.'s) will be denoted $L(\,\cdot\,,\,\cdot\,)$,

$$L(F,\, G) = \inf\{\epsilon :\; G(x-\epsilon) - \epsilon \leqslant F(x) \leqslant G(x+\epsilon) + \epsilon \;\forall x \in \mathbb{R}\}.$$

To each pair of d.f.'s $(F,\, H)$ put into correspondence the set $\mathcal{M}(F|H)$ containing all d.f.'s $Q(x)$ with $Q(0) = 0$ providing the representation of the characteristic function (ch.f.) corresponding to the d.f. F as a power mixture of ch.f.'s corresponding to the d.f. H:

$$\int_{-\infty}^{\infty} e^{itx} dF(x) = \int_{0}^{\infty} h^{x}(t) dQ(x), \text{ where } h(t) = \int_{-\infty}^{\infty} e^{itx} dH(x), \quad t \in \mathbb{R}.$$

Everywhere in what follows the convergence will be meant as $n \to \infty$.

Lemma 1. *Assume that there exist $k_n \in \mathbb{N}$, $n \geqslant 1$, and a d.f. $H(x)$ such that*

$$\mathsf{P}\big(S_{n,k_n} < x\big) \Longrightarrow H(x).$$

Assume that $N_n \to \infty$ in probability. Then the convergence

$$\mathsf{P}\big(S_{n,N_n} < x\big) \Longrightarrow F(x) \tag{3}$$

of the distributions of random sums to a d.f. $F(x)$ takes place if and only if there exists a weakly compact sequence of d.f.'s $\{Q_n^*(x)\}_{n\geqslant 1}$ such that

(i) $Q_n^*(x) \in \mathcal{M}(F|H)$, $n = 1, 2, \ldots$,

(ii) $L(Q_n^*, Q_n) \longrightarrow 0$,

where $Q_n(x) = \mathsf{P}(N_n < k_n x)$, $x \in \mathbb{R}$.

This statement is a particular case of theorem 4.2.1 in [3].

Let a function $H(x; y)$ be defined on $\mathbb{R} \times \mathbb{R}$. Assume that $H(x; y)$ is measurable with respect to y for each fixed $x \in \mathbb{R}$ and is a d.f. as the function of x for each fixed $y \in \mathbb{R}$. Let $\mathcal{Q}$ be a family of d.f.'s. Denote

$$\mathcal{F} = \left\{ F(x) = \int_{-\infty}^{\infty} H(x; y)\, dQ(y),\; x \in \mathbb{R} :\; Q \in \mathcal{Q} \right\}.$$

The family $\mathcal{F}$ is called *identifiable*, if the equality

$$\int_{-\infty}^{\infty} H(x; y)\, dQ_1(y) = \int_{-\infty}^{\infty} H(x; y)\, dQ_2(y), \qquad x \in \mathbb{R},$$

with $Q_1 \in \mathcal{Q},\; Q_2 \in \mathcal{Q}$ implies $Q_1(y) \equiv Q_2(y)$.

The standard normal d.f. will be denoted $\Phi(x)$. It is well known that general scale-location mixtures of normal laws are not identifiable. However, the family $\mathcal{F}_\Phi$ of one-parameter *variance-mean* mixtures of normal laws

$$\mathcal{F}_\Phi = \Big\{ F(x) = \int_0^\infty \Phi\Big(\frac{x-\alpha y}{\sigma\sqrt{y}}\Big) dQ(y),\ x \in \mathbb{R} :\ Q \in \mathcal{Q} \Big\}$$

with fixed $\alpha \in \mathbb{R}$ and $\sigma > 0$ turns out to be identifiable, since the family of d.f.'s $\big\{\Phi\big((x-\alpha y)/(\sigma\sqrt{y})\big) :\ y > 0\big\}$ is additively closed (see [4], [5]).

With the account of the identifiability of $\mathcal{F}_\Phi$, lemma 1 implies

THEOREM 1. *Let there exist $k_n \in \mathbb{N}$, $n \geqslant 1$, and $\alpha \in \mathbb{R}$ such that*

$$\mathsf{P}\big(S_{n,k_n} < x\big) \Longrightarrow \Phi(x-\alpha). \tag{4}$$

Assume that $N_n \to \infty$ in probability. Then the convergence (3) *of the distributions of random sums to a d.f. $F(x)$ takes place if and only if there exists a d.f. $Q(x)$ such that $Q(0) = 0$,*

$$F(x) = \int_0^\infty \Phi\Big(\frac{x-\alpha z}{\sqrt{z}}\Big) dQ(z) \quad \textit{and} \quad \mathsf{P}(N_n < xk_n) \Longrightarrow Q(x).$$

REMARK 1. Condition (4) holds in the following situation. Assume that $0 < \mathsf{D}X_{n,j} < \infty$. Also assume that the r.v.'s $X_{n,j}$ can be represented as

$$X_{n,j} = X^*_{n,j} + \alpha_n,$$

where $\alpha_n \in \mathbb{R}$, $\mathsf{E}X^*_{n,j} = 0$, $0 < \mathsf{D}X^*_{n,j} = \sigma_n^2 < \infty$, so that $\mathsf{E}X_{n,1} = \alpha_n$ and $\mathsf{D}X_{n,1} = \sigma_n^2$. Let $\alpha_n k_n \to \alpha$ and $k_n\sigma_n^2 \to 1$. Then, as is known (see, e.g., [6]), relation (4) holds if and only if the Lindeberg condition holds: for any $\varepsilon > 0$

$$\lim_{n\to\infty} k_n \mathsf{E}(X^*_{n,1})^2 \mathbb{I}(|X^*_{n,1}| \geqslant \varepsilon) = 0,$$

(here $\mathbb{I}(A)$ is the indicator function of a set A).

COROLLARY 1. *Let there exist $k_n \in \mathbb{N}$, $n \geqslant 1$, and $\alpha \in \mathbb{R}$ such that* (4) *holds. Assume that $N_n \to \infty$ in probability. Then the distributions of random sums converge to a generalized variance gamma distribution*:

$$\mathsf{P}\big(S_{n,N_n} < x\big) \Longrightarrow W(x;\alpha,\nu,\kappa,\delta)$$

if and only if

$$\mathsf{P}(N_n < xk_n) \Longrightarrow F(x;\nu,\kappa,\delta), \tag{5}$$

where $F(x;\nu,\kappa,\delta)$ is a GG-d.f. corresponding to the density $f(x;\nu,\kappa,\delta)$ (*see* (1)).

We also give some simple scheme which allows to easily construct r.v.'s satisfying (5) and discuss convergence rate estimates in corollary 1.

This research was supported by the Russian Foundation for Basic Research (projects 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a), and by the Ministry for Education and Science of Russian Federation (state contract 16.740.11.0133).

References

1. *Stacy E. W.* A generalization of the gamma distribution // Annals of Mathematical Statistics, 1962. Vol. 33. P. 1187–1192.

2. *Korolev V. Yu., Sokolov I. A.* Skew Student distributions, variance gamma distributions and their generalizations as asymptotic approximations // Informatics and its Applications, 2012. Vol. 6. No. 1. P. 2–10.

3. *Gnedenko B. V., Korolev V. Yu.* Random Summation: Limit Theorems and Applications. – Boca Raton: CRC Press, 1996.

4. *Teicher H.* Identifiability of mixtures // Ann. Math. Stat., 1961. Vol. 32. P. 244–248.

5. *Korolev V. Yu.* Generalized hyperbolic laws as limit distributions for random sums // Probability Theory Appl., 2012, to appear.

6. *Gnedenko B. V., Kolmogorov A. N.* Limit Distributions for Sums of Independent Random Variables. Reading, MA: Addison–Wesley, 1954.

Reconstruction of tomographic images using Fourier-Wavelet decomposition

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Tomographic techniques of image reconstruction are widely used in different fields of science and technology. The main mathematical tool in many tomographic experiments is the Radon transform:

$$Rf(\varphi, s) = \int\limits_{L_{\varphi,s}} f(x,y)dl,\ s \in \mathbb{R}, \varphi \in [0, 2\pi),$$

where $L_{\varphi,s}$ is the line defined by the angle φ and the distance s. In real tomographic experiments one always have to deal with the noisy measurements. So we consider the following model:

$$X_{u,v} = (Rf)_{u,v} + \varepsilon_{u,v},\quad u = 1, \dots, 2^J,\ v = 1, \dots, 2^J. \tag{1}$$

Here J is some positive integer number, $X_{u,v}$ are the observed data, and $\varepsilon_{u,v}$ are independent normal variables with zero mean and variance equal to σ^2.

Nonlinear wavelet methods of de-noising are becoming more and more popular because of their ability to capture local singularities of images (see [1]). One possibility is to use the following approximate image decomposition (see [2]):

$$f = \sum_{n=0}^{2^J-1} \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \beta_{n,j,k} \langle Rf_n, \psi_{j,k} \rangle u_{n,j,k},$$

where $\{\psi_{j,k}\}$ is a wavelet basis generated by a certain mother wavelet ψ, $\beta_{n,j,k}$ are normalization constants, Rf_n are Fourier harmonics of Rf, and $\{u_{n,j,k}\}$ is a corresponding "vaguelette" basis, which appears to be stable if mother wavelet ψ satisfies certain regularity conditions (see [2]).

To filter out the noise we use thresholding method with soft-thresholding function $\rho_{T_j}(x) = \operatorname{sgn}(x)\left(|x| - T_j\right)_+$, and obtain an estimate of the image:

$$\hat{f} = \sum_{n=0}^{2^J-1} \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \beta_{n,j,k} \rho_{T_j}(Y_{n,j,k}) u_{n,j,k}, \tag{2}$$

where $Y_{n,j,k}$ are noisy decomposition coefficients of the image. Here we use individual threshold $T_j = \sqrt{2 \ln 2^{j+J}}\sigma$ for each decomposition level j. This threshold is called "universal" (see [2]).

Risk (average mean squared error) of soft thresholding method is defined as

$$r_J = \sum_{n=0}^{2^J-1} \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \beta_{n,j,k}^2 \mathrm{E}(2^{J/2} \langle Rf_n, \psi_{j,k} \rangle - \rho_{T_j}(Y_{n,j,k}))^2. \tag{3}$$

This expression contains unknown values $\langle Rf_n, \psi_{j,k} \rangle$, so it cannot be calculated and has to be estimated. Following D. Donoho and I. Johnstone (see [3]) we propose to use SURE estimate

$$\hat{r}_J = \sum_{n=0}^{2^J-1} \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \beta_{n,j,k}^2 F_{T_j}(Y_{n,j,k}), \tag{4}$$

where $F_{T_j}(x) = (x^2 - \sigma^2)\mathrm{I}(|x| \le T_j) + (\sigma^2 + T_j^2)\mathrm{I}(|x| > T_j)$. This estimate is unbiased, i.e. $\mathrm{E}\hat{r}_J = r_J$. We prove that under certain conditions it is also asymptotically normal. The following theorem holds.

Theorem. *Let mother wavelet ψ have sufficient number of vanishing moments and satisfy certain conditions, which ensure that basis $\{u_{n,j,k}\}$ is stable (see [2]). Let f have compact support and be Lipschitz continuous of order $\gamma > 0$. Then*

$$\frac{\hat{r}_J - r_J}{\sigma^2 \beta_{0,0,0}^2 \sqrt{2/7}\; 2^{2J}} \Longrightarrow N(0,1) \quad \text{as} \quad J \to \infty. \tag{5}$$

In (5) we do not use traditional normalization which involves variance of $\hat{r}_J$, because this variance depends on the unknown values $\langle Rf_n, \psi_{j,k}\rangle$. Proposed normalization allows to construct asymptotic confidence intervals for r_J.

Acknowledgements: this work is supported by RFBR (grants 11–07–00112a, 11–01–00515a and 11-01-12-26-ofi-m)

References

1. Abramovich F., Silverman B. W. Wavelet Decomposition Approaches to Statistical Inverse Problems. *Biometrika*, 1998. Vol. 85. No. 1. P. 115–129.

2. Kudryavtsev A.A., Shestakov O.V. Reconstruction of tomographic images by means of thresholding coefficients of the expansion of projection data by orthogonal basis. *T-Comm – Telecommunications and Transport*, 2012. No. 3. P. 63–68. (in Russian)

3. Donoho D., Johnstone I. Adapting to Unknown Smoothness via Wavelet Shrinkage. *J. Amer. Stat. Assoc.*, 1995. Vol. 90. P. 1200–1224.

Multistate Markov models penalized estimates

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Many reliability processes are described in terms of Markov multistate models. The number of states and definitions of states depend on the problem at hand. Often transitions between the states are made at random times with intensities, depending on different covariates and, possible, on time. If the matrix of transition intensities is given, then one can calculate probability that the system made transition from state S_i to state S_j for the given time. This is direct problem, which can be solved using the set of Kolmogorov equations.

Identification of transitions intensities matrix is an inverse problem and depending on the characteristics of the data can be solved with different results. If the data present the sequence of states combined with times, when the transitions between the states were made $\{S_1, t_1, ..., S_n, t_n\}$, then the maximum likelihood estimates for transitions intensities are $\hat{\lambda}_{ij} = n_{ij}/T_i$. Here n_{ij} denotes the observed number of transitions from the state S_i to the stateS_j, T_i is the total time spent by all objects at the stateS_i.

Often the states of the objects are observed not at the moments of the transitions but at the times, not dependent on the states as in the case of regular investigations. In this case one knows the states of the object at the given times but does not know the moments of transitions. This is a case of interval censored observations [1]. To obtain the estimates for the transition intensity matrix one is to maximize the likelihood function and this problem is ill-posed and instable [2]. It is proposed in the report to stabilize the maximum likelihood estimates by maximization of posterior probability for the transitions intensity matrix. This leads to penalized likelihood maximization with penalty term, derived from the natural condition on the time, which the process stays in the selected state.

References

1. D. Commenges. Inference for multi-state models from interval-censored data. *Stat Methods Med Res.*, 2002, vol. 11, No. 2, h. 167–182.

2. H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems, volume 375 of Mathematics and its Applications*, 1996. Kluwer Academic Publishers Group, Dordrecht.

On the Gaussian asymptotics of the binomial distributions

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Let X have a two-point distribution $\mathbf{P}(X{=}a){=}q$, $\mathbf{P}(X{=}d){=}p$, $p+q=1$, where $a<0<d$, $\mathbf{E}X=0$, $\mathbf{E}X^2=1$. Denote its distribution function by $F(x)$. Define $\alpha_3(p)=\mathbf{E}X^3$, $\beta_3(p)=\mathbf{E}|X|^3$, $\tau=\left(\frac{6}{\beta_3(p)n}\right)^{1/3}$, $\delta_{n,p}(x)=F^{*n}(x)-\Phi(x/\sqrt{n})$, $\Delta_n(p)=\sup_x|\delta_{n,p}(x)|$, where $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\,dt$. Evidently, $\Delta_n(p)$ coincides with the distance in uniform metric between the standardized binomial distribution and the standard normal distribution.

Note that $\Delta_n(p)$ is attained at a discontinuity point of the convolution F^{*n}. Denote this point by x_0. Four cases are possible: $\delta_n(p,x_0+)=\Delta_n(p)$, $\delta_n(p,x_0-)=\Delta_n(p)$, $\delta_n(p,x_0+)=-\Delta_n(p)$, $\delta_n(p,x_0+)=-\Delta_n(p)$. Consider, for instance, the case $\delta_n(p,x_0+)=\Delta_n(p)$. Define

$$A(p,n,x)=\frac{\alpha_3(p)}{3!\,2\pi\sqrt{n}}\int_{|u|\leqslant\tau\sqrt{n-1}}u^2e^{-u^2/2}\,\frac{\sin y}{y}\Big|_{y=\frac{uh}{2\sqrt{n-1}}}\sin\left(\frac{ux}{\sqrt{n-1}}\right)du,$$

$$B(p,n,x_0)=\frac{1}{h}\int_{x_0}^{x_0+h}\Big(\Phi(u/\sqrt{n})-\Phi(x_0/\sqrt{n})\Big)\,du,$$

where h is an arbitrary positive number. Note that $0<B(p,n,x_0)<\frac{h}{2\sqrt{2\pi n}}$.

Let $P(x)$ be the uniform distribution on $[-h/2,h/2]$.

Theorem 1. *For each discontinuity point x_0 of the function F^{*n} the following equality holds,*

$$\delta_{n,p}(x_0+) = (P * \delta_{n,p})(x_0 + h/2) + B(n, p, x_0).$$

Let us introduce the condition

$$\frac{4}{n} \leqslant p \leqslant 0.5, \quad n \geqslant 200. \tag{1}$$

Further, let $K_1(p,n)$, $K_2(p,n)$ and $K_3(p,n)$ be the functions from [1].

Theorem 2. *Let $n \geqslant 200$. Then for each x,*

$$(P * \delta_{n,p})(x) = A(p, n, x) + R(p, n),$$

where $|R(p,n)| \leqslant \sum_{i=1}^{3} K_i(p,n)$, if the condition (1) is fulfilled. In addition, for every fixed $p \in (0; 0.5]$, the sequence $R_0(p,n) = \frac{\sqrt{n}}{\beta_3(p)} \sum_{i=1}^{3} K_i(p,n)$ is $O(1/\sqrt{n})$, decreasing in $n \geqslant \max\{200; 4/p\}$.

Corollary 1. *Let $n \geqslant 200$. Then for each discontinuity point x_0 of the function F^{*n},*

$$\delta_{n,p}(x_0+) = A(p, n, x_0 + h/2) + B(n, p, x_0) + R(p, n),$$

where $R(p,n)$ is the function from Theorem 2.

Remind the Esseen function $\mathcal{E}(p) = \frac{2-p}{3\sqrt{2\pi}\left[p^2+(1-p)^2\right]}$ (see [1], [2]), and the Esseen constant $C_E \equiv \frac{\sqrt{10}+3}{6\sqrt{2\pi}} = 0.409732\ldots$.

Using the inequality $|A(p, n, x_0 + h/2) + B(p, n, x_0)| \leqslant \frac{\alpha_3(p)}{6\sqrt{2\pi n}} + \frac{1}{2\sqrt{pq2\pi n}} = \frac{\beta_3(p)}{\sqrt{n}}\,\mathcal{E}(p)$, we obtain

Theorem 3. *Let $\frac{4}{n} \leqslant p \leqslant 0.5$, $n \geqslant 1600$. Then*

$$\Delta_n(p) \leqslant \frac{\beta_3(p)}{\sqrt{n}}\,\mathcal{E}(p) + |R(p,n)|,$$

where $R(p,n)$, being the function from Theorem 2, satisfies the inequality $\frac{\sqrt{n}}{\beta_3(p)}\,|R(p,n)| < 0.4138 - C_E = 0.004067\ldots$.

Theorem 4.

$$C_0 \equiv \sup_{n,\,p} \frac{\sqrt{n}}{\beta_3(p)}\,\Delta_n(p) < 0.4138. \tag{2}$$

Note that from the bound $\Delta_n(p) \leqslant \frac{0.3328}{\sqrt{n}}\Big(\beta(p) + 0.429\Big)$ proved in [3] for arbitrary i.i.d.r.v's we obtain that in the case $0 < p \leqslant 1/400$ the following bound holds, $\frac{\sqrt{n}}{\beta_3(p)}\,\Delta_n(p) < 0.34$.

As to the case $1 \leqslant n \leqslant 1600$, we show by using a computer that $\max\limits_{1 \leqslant n < 1600} \max\limits_{p \in (0,0.5]} \frac{\sqrt{n}}{\beta(p)} \Delta_n(p) < C_E$.

Acknowledgements. This work was fulfilled under the partial support by grants: Siberian Branch of RAS 56, Far-Eastern Branch of RAS 12-II-SO-01-002, 12-I-OMN-01.

References

1. S. V. Nagaev, V. I. Chebotarev. On the bound of proximity of the binomial distribution to the normal one. *Theory of Probability and its Applications*, 2012, vol. 56, No. 2, p. 213-239.

2. C.-G. Esseen. A moment inequality with an application to the central limit theorem. *Scand. Aktuarietidskr. J.*, 1956, vol. 39. p. 160–170.

3. I. G. Shevtsova. On the absolute constants in the BerryEsseen type inequalities for identically distributed summands. *ArXiv: 1111.6554v1*, 2011.

Asymptotic properties of grid method estimators in the normal mixture separation problems

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The problem of separation of mixtures of probability distributions is traditionally reduced to the problem of parameter estimation within a certain mathematical model framework. In this case the number of estimated parameters is relatively small. However, in cluster analysis and non-parametric density estimation problems it is required to construct an estimate of mixing distribution function based on the corresponding mixture realizations.

EM-algorithm and its modifications (see [1]) are often used to find maximum likelihood estimates in problems of separation of mixtures. In this case, the estimate of the mixing distribution is concentrated at a fixed number of points (atoms). However, the classical EM-algorithm has some major disadvantages. For example, the number of atoms and the starting point in maximization procedure should be specified explicitly. The EM-algorithm is also very sensitive to the choice of the initial approximation: it is possible to obtain quite different results using different starting points on the same set of mixture realizations.

To overcome the above disadvantages, in [1–3] the so-called *grid* methods of separation of mixtures were proposed. These methods are based on the assumption that the mixing distribution estimate should be sought in the class of distributions concentrated on the set of fixed points (the grid). In this communication it is shown that grid method estimators in normal mixtures decomposition problems are consistent within some subclasses of mixtures. A necessary condition of consistency is that the grid size should be a certain function of the sample size.

This fact proves that grid methods provide reasonable estimates and can be used in various practical problems.

This research was supported by the Russian Foundation for Basic Research (projects 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a), and by the Ministry for Education and Science of Russian Federation (state contract 16.740.11.0133).

References

1. V. Yu. Korolev. *Veroyatnostno-statisticheskii analiz khaoticheskikh protsessov s pomoshchyu smeshannykh gaussovskikh modelei. Dekompozitsiya volatil'nosti finansovykh indeksov i turbulentnoi plazmy*(Probabilistic and Statistical Analysis of Chaotic Processes with Gaussian Mixture Models. Decomposing the Volatility of Financial Indices and Turbulent Plasma), 2007, Moscow: Inst. Probl. Informatiki.

2. V. Yu. Korolev, A. L. Nazarov. Separating mixtures of probability distributions with the grid method of moments and the grid maximal likelihood method. *Autom. Remote Control*, 2010, Vol. 71, no. 3, p. 455-472.

3. V. Yu. Korolev, E. V. Nepomnyashchiy, A. G. Rybal'chenko, A. V. Vinogradova. Network methods of separation of mixtures of probability distributions and their application to the decomposition of volatility indexes *Inform. Primen.*, 2008, Vol. 2, p. 3-18.

4. A. W. van der Vaart. *Asymptotic Statistics (Cambridge Series in Statistical and Probabilistic Mathematics)*, 2000, Cambridge University Press.

Lower bounds for the stability of normal mixture models with respect to perturbations of mixing distribution

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Many popular stochastic models are based on the usage of mixtures of probability distributions. For example, these models are used in modeling the evolution of prices of financial instruments or turbulent plasmas. They are also used in solving pattern recognition problems. Examples of random processes whose one-dimensional distributions are mixtures of normal distributions can be found in [1] and the references therein.

We consider the problem of stability of a mixture of probability distributions with respect to perturbations of mixing distribution. The first results of this type for the special case, a simple Tukey contamination model [2], was obtained in [3]. A solution of this problem for another special case can be found in [4] where estimates of the distance between the normal distribution and a scale mixture under certain conditions imposed on the mixing distribution were presented. Upper bounds in this stability problem were considered in [5].

The aim of this study is to obtain the lower bounds in mixture stability problem. Inequality estimating the distance between two mixing distributions through the closeness of the corresponding mixtures is presented. Existence theorem for stability estimates is proved for subclasses of scale and shift mixtures of normal distributions. The estimate for the class of shift mixtures of normal distributions is obtained in an explicit form. It is also demonstrated that the presented results cannot be radically improved without additional assumptions.

The obtained results are important for the study of the asymptotic properties of estimates in the problem of separation of mixtures by the grid methods [3].

This research was supported by the Russian Foundation for Basic Research (projects 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a), and by the Ministry for Education and Science of Russian Federation (state contract 16.740.11.0133).

References

1. V. Yu. Korolev. *Veroyatnostno-statisticheskii analiz khaoticheskikh protsessov s pomoshch'yu smeshannykh gaussovskikh modelei. Dekompozitsiya volatil'nosti finansovykh indeksov i turbulentnoi plazmy* (Probabilistic and Statistical Analysis of Chaotic Processes with Gaussian Mixture Models. Decomposing the Volatility of Financial Indices and Turbulent Plasma), 2007, Moscow: Inst. Probl. Informatiki.

2. J. W. Tukey. A survey of sampling from contaminated distributions. *Contributions to Probability and Statistics Essays in Honor of Harold Hotelling*, 1960, p. 448–485.

3. V. Yu. Korolev, A. L. Nazarov. Separating mixtures of probability distributions with the grid method of moments and the grid maximal likelihood method. *Autom. Remote Control*, 2010, Vol. 71, no. 3, p. 455–472.

4. P. Hall, A. L. Nazarov. On measures of the distance of a mixture from its parent distribution. *Stochastic Processes and their Applications*, 1979, Vol. 8, p. 357–365.

5. A. L. Nazarov. On the stability of probability distribution mixtures with respect to mixing distributions perturbations. – In: *Statisticheskie metodi ocenivaniya i proverki gipotez*, Perm: Perm State University Publishing House, 2010, Vol. 22, p. 154–172.

Estimates of the accuracy of the approximation to the distributions of the negative binomial random sums

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Let $X_1, X_2, \ldots$ be independent random variables with the common distribution function $F(x) = \mathsf{P}(X_1 < x)$ and satisfying the conditions

$$\mathsf{E}X_1 = 0, \quad \sigma^2 = \mathsf{E}X_1^2, \quad \beta_{2+\delta} = \mathsf{E}|X_1|^{2+\delta} < \infty,$$

for some $0 < \delta \leq 1$. Consider *the negative binomial random sum*

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \quad \left(\sum_{i=1}^{0}(\cdot) \equiv 0\right),$$

where the random variable $N(t)$ has the negative binomial distribution with parameters $r > 0$ $p = (1+t)^{-1}$, $t > 0$:

$$\mathsf{P}(N(t) = k) = \frac{\Gamma(r+k)}{k! \cdot \Gamma(r)} p^r (1-p)^k, \quad k = 0, 1, 2, \ldots.$$

Assume that the random variables $N(t), X_1, X_2, \ldots$ are independent for each $t > 0$.

As is well known, the negative binomial with parameters $r > 0$ and $p \in (0, 1)$ is the mixed Poisson distribution with the mixing gamma-distribution $G_{r,s}(x)$ with shape parameter $r > 0$ and scale parameter $s = p/(1-p) = 1/t > 0$.

The random sum $S(t)$ is also called a *mixed Poisson random sum* and its distribution is called *compound mixed Poisson.*

In the case $r > \delta/2$ the convergence rate estimate in the limit theorem for the negative binomial random sums to the scale mixture of normal law is known. Under the above conditions on the moments of random variable X_1 and condition $r > \delta/2$ for each $t > 0$ the following analog of the Berry–Esseen inequality holds

$$\rho_t \equiv \sup_x \left| \mathsf{P}(S(t) < x\sigma\sqrt{rt}) - \int_0^{+\infty} \Phi\left(\frac{x}{\sqrt{\lambda}}\right) dG_{r,r}(\lambda) \right| \leq C(r;\delta) \frac{\beta_{2+\delta}}{\sigma^{2+\delta} t^{\delta/2}},$$

where $C(r;\delta) = C(\delta)\Gamma(r-\frac{\delta}{2})/\Gamma(r)$, and the constant $C(\delta)$ is the same as in the Berry–Esseen inequality for Poisson random sums. In particular $C(1) \leq 0.3041$ (see [1]).

This statement was first proved in the paper [2] with a slightly worse constant. The best known upper bounds for $C(r;\delta)$ are obtained by V. Korolev and I. Shevtsova in [1]. The non-trivial lower bounds for $C(r;\delta)$ were find by Yu. Nefedova in [3]. Moreover, it was shown that obtained minorants $C(r;\delta)$ are positive for all $\delta \in (0, 1]$ and $r > \delta/2$, so we can conclude that the order $O(t^{-\delta/2})$ of convergence rate is correct as $t \to \infty$ for the uniform in F estimates ρ_t.

Here we construct the new, practically applicable estimates of the accuracy of the approximation to the distributions of the negative binomial random sums when the parameter $r > 0$ of the negative binomial distribution satisfies the following condition: $r \leq \delta/2$.

We will show that in the case $r < \delta/2$ the convergence rate estimate has the order $O(t^{-r})$ and in the case $r = \delta/2$ the order is $O(t^{-\delta/2}\ln(t))$, $t \to \infty$. In both cases, we prove that obtained order is correct.

This research was supported by the Russian Foundation for Basic Research (projects 12-01-31125, 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a) and by the Ministry for Education and Science of Russia (project MK–2256.2012.1).

References

1. V. Yu. Korolev, I. G. Shevtsova. An improvement of the Berry–Esseen inequality with applications to Poisson and mixed Poisson random sums. *Scandinavian Actuarial Journal*, 2012, No. 2, p. 81–105. Available online since 04 June 2010.

2. S. V. Gavrilenko, V. Yu. Korolev. Convergence rate estimates for mixed Poisson random sums. – In: *Systems and Means of Informatics*, Special Issue. Publishing House of the Institute for Informatics Problems, Russian Academy of Sciences, Moscow, 2006, p. 248–257 (in Russian).

3. Yu. S. Nefedova. Estimates of the rate of convergence in the limit theorem for negative binomial random sums *Statistical Methods of Estimation and Testing Hypotheses.* – Perm: Rerm State University Publishing House, 2011. P. 46–61 (in Russian).

Fractional Levy process as a limit one in infinite source renewal model of teletraffic

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Recently (see [1]) we have proposed a new model of cumulative teletraffic in the following form. Let $(B_H(t), t \geq 0)$ be fractional Brownian motion with Hurst parameter H, $(L^1_\alpha(t), t \geq 0)$, $(L^2_\alpha(t), t \geq 0)$ be α-stable subordinators, $0 < \alpha \leq 1$, and B_H, L^1_α and L^2_α are independent. Consider the new process

$$X(t) := \begin{cases} B_H(L^1_\alpha(t)) & , \quad t \geq 0, \\ -B_H(L^2_\alpha(t)) & , \quad t < 0, \end{cases}$$

This process capture both the properties of long range dependence and heavy tails of distributions. Moreover the above process X is self-similar process with Hurst parameter $H_1 = H \cdot \alpha$ We have used it for estimation of buffer overflow probability.

In our present report we show how to get this process as a limit one in infinite source renewal model of teletraffic. Our result is analog of the result from [2]. First we describe the standard infinite source Poisson model following paper [2]. Let $(\Gamma_j, \;\; -\infty < j < \infty)$ be the point process generated by homogeneous Poisson process in R^1 with parameter λ, labeled so that $\Gamma_0 < 0 < \Gamma_1$ and hence $\{\Gamma_0, \Gamma_1, (\Gamma_{j+1} - \Gamma_j, \;\; j \neq 0)\}$ are i.i.d. exponentially distributed random variables with parameter λ. We imagine that a communication system has an infinite number of nodes or sources, and at the time Γ_j a connection is made and some node begins a transmission at constant rate to server. In what follows this constant rate is taken to be unity. The lengths of transmissions are random variables $(X_j, \;\; j \in Z)$ which are i.i.d. and independent of (Γ_j). We assume that they have the common distribution function $F(x)$ such that

$$\bar{F}(x) := Pr(X_j > x) = x^{-\beta} L(x), \quad x > 0, \tag{1}$$

where $L(x)$ is slowly varying function as $x \to \infty$, $1 < \beta < 2$. In this case there exists the finite expectation $\mu = E(X_j)$. Also define the quantile function

$$b(t) := \inf(x : 1/\bar{F}(x) \geq t).$$

Function $b(t)$ is non-decreasing and regular varying with index $1/\beta$.

$N(t)$ denotes the number of active sources at time t:

$$N(t) = \sum_{k=-\infty}^{\infty} \mathbf{1}_{\{\Gamma_k \leq t < \Gamma_k + X_k\}}.$$

Then $A(t) = \int_0^t N(s)ds$ is the *total cumulative input* in $[0,t]$.

Next we consider infinite source Poisson model indexed by with scaling parameter $T > 0$ such that the intensity $\lambda = \lambda(T)$ goes to infinity as $T \to \infty$. $\lambda = \lambda(T)$ will be referred to as the *connection rate*. In what follows we are interested in limit behavior of random process $(B_T(t) := A(T \cdot t), \quad t \geq 0)$ as $T \to \infty$. In paper [2] the following result was proved.

Theorem 1. *Assume that the process $B_T(t)$ satisfies the Fast Growth Condition (FGC):*

$$\frac{b(\lambda(T) \cdot T)}{T} \underset{T\to\infty}{\longrightarrow} \infty \ . \tag{2}$$

Then the following convergence

$$\frac{B_T(t) - T \cdot \lambda(T) \cdot \mu \cdot t}{(\lambda(T) \cdot T^3 \cdot \bar{F}(T) \cdot \sigma^2)^{1/2}} \underset{T\to\infty}{\Longrightarrow} B_H(t)$$

holds, where $(B_H(t), \quad t \geq 0)$ is a standard fractional Brownian motion, $H = (3-\beta)/2$ and

$$\sigma^2 = \frac{1}{3-\beta}\left[\frac{\beta}{2-\beta} + \frac{2}{\mu}\right].$$

Now we replace the sequence $\{\Gamma_j\}$ by the sequence of positive i.i.d.r.v. $\{Y_j\}$ (renewal process!) whose tails have the form (1) with index $\alpha_1 < 1$ and consider the analog $\tilde{B}_T(t)$ of process $B_T(t)$. Our main result is the following

Theorem 2. *Assume that the process $\tilde{B}_T(t)$ satisfies the FGC. Then after some normalization of the process $\tilde{B}_T(t)$ we have the process $X(t)$ as the limit one*

References

1. C. De Nikola, Y. S. Khokhlov, M. Pagano and O. I. Sidorova. Fractional Levy motion with dependent increments and its application to network traffic modeling, *Informatics and Applications*, 2012 (in print).

2. Mikosch Th., Resnick S., Rootzen H., Stegeman A. Is network traffic approximated by stable Levy motion or fractional Brownian motion? *Ann. Appl. Probab.*, 2002, vol. 12, No. 1, p. 23–68.

New techniques for magnetoencephalogram and myogram signal processing

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As part of brain areas localization task set before mathematical statistics department we developed a new technique for MEG-signal processing Among other issues this includes detection of primary motor cortex and nonrenewable brain regions. We investigated statistical and stochastic characteristics of our signals and revealed noise non-normality and non-stationary.

By using this approach we developed sustained algorithms for fully automated signal processing and for identification of reference points within motor cortex localization problem.

These reference points are engaged for averaging of the source signals and creation of brain activity template, which is used further to solve the inverse problem – functional brain mapping. To achieve adequate accuracy of localization it is necessary to identify reference points with a detection error up to several record units (sampling frequency is 1 kHz).

It should be noted that typically signal comprise few million units (references). The algorithms are based on the stochastic signal characteristics, some of them employed wavelet analysis (see [1]) Our method is able to deal effectively with noise-contaminated signals (up to 5% of each MEG-record is formed usually by useful signal) and non-stationary signals.

Some of the algorithms are implemented by Moscow MEG-center and allowed them to improve accuracy of primary motor cortex localization for each patient (see [2]).

Developed algorithms are important to establishment of clinical procedure for neurosurgical practice. This method is of particular value for patients with various brain pathologies and hence skewed brain topography.

Magnetoencephalography can be used as a diagnostic technique for focal brain lesions as well as brain pathological functionality.

Constantly working on improving the localization accuracy for various brain areas we enhance our method and develop new algorithms. This is a challenge for many researches around the world since MEG-signal processing give us an opportunity for non-invasive functional brain studying (see [2,3]).

References

1. T.V.Zakharova, O.V.Shestakov. *Wavelet analysis and its application.* Max Press, 2009 [in Russian].

2. T.V.Zakharova, S.Y.Nikiforov, M.B.Goncharenko, M.A. Dranitsyna, G.A.Klimov, M.S.Khaziakhmetov, N.V.Chayanov. *Signal processing*

methods for the localization of nonrenewable brain regions. // Informatics and Applications(article in press).

3. M.Hamalainen, R.Hari, R.J.Ilmoniemi, J.Knuutila, O.V.Lounasmaa. *Magnetoencephalography – theory, instrumentation, and applications to noninvasive studies of the working human brain* // Rev. Mod. Phys., 1993, vol. 65, p. 413–497.

Asymptotically normal class of estimators of parameters from incomplete survival data

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Let X - be the random variable (r.v.), lifetime of individual with survival function $1 - F(x;\theta)$, depending on unknown parameter θ and density $f(x;\theta)$, $\theta \in \Theta \subseteq R^1$. Assume that the r.v. X subject to random censoring from both sides by r.v.-s L and Y with distribution functions (d.f.-s) K and G and densities k and g respectively, which are independent of θ.

Let $\{(L_i, X_i, Y_i), i \geqslant 1\}$ - be a sequence of independent replicas of vector (L, X, Y) with independent components. Observations is available the sample $\{\widetilde{Z}_i = (Z_i; \Delta_i^{(0)}, \Delta_i^{(1)}, \Delta_i^{(2)}), 1 \leqslant i \leqslant n\} = \mathbb{V}^{(n)}$, where $Z_i = \max(L_i, \min(X_i, Y_i))$, $\Delta_i^{(0)} = I(\min(X_i, Y_i) < L_i)$, $\Delta_i^{(1)} = I(L_i \leqslant X_i \leqslant Y_i)$, $\Delta_i^{(2)} = I(L_i \leqslant Y_i < X_i)$ and $I(A)$ is an indicator of event A. We denote $\widetilde{Z}^{(n)} = (\widetilde{Z}_1, \widetilde{Z}_2, ..., \widetilde{Z}_n)$ and let $\{\mathcal{Y}^{(n)}, \mathcal{U}^{(n)}, Q_\theta^{(n)}\}$ is a sequence of statistical experiments generated by observations $\widetilde{Z}^{(n)}$, where $\mathcal{Y}^{(n)} = \{\mathcal{X} \times \{0,1\}^{(3)}\}^{(n)}$, $\mathcal{U}^{(n)} = \sigma(\mathcal{Y}^{(n)})$ and $Q_\theta^{(n)}$ is the distribution on $(\mathcal{Y}^{(n)}, \mathcal{U}^{(n)})$ with one-dimensional distribution $Q_\theta(x, y^{(0)}, y^{(1)}, y^{(2)}) = P(Z_i \leqslant x, \Delta_i^{(0)} = y^{(0)}, \Delta_i^{(1)} = y^{(1)}, \Delta_i^{(2)} = y^{(2)})$, $y^{(m)} \in \{0,1\}$, $m = 0,1,2$. Let $\varepsilon_{y(m)}$ is a counting measure concentrated at the point $y^{(m)}$ and $d\nu(\widetilde{Z}_i) = \varepsilon_{y_i^{(m)}} \times dZ_i, i = \overline{1,n}$. Then the distribution of $Q_\theta^{(n)}$ is absolutely continuous with respect to $\nu^{(n)}(\widetilde{Z}^{(n)}) = \nu(\widetilde{Z}_1) \times ... \times \nu(\widetilde{Z}_n)$ and have the density for any $\theta \in \Theta$ defined on $\mathcal{Y}^{(n)}$ as

$$p_n(\widetilde{Z}^{(n)};\theta) = \frac{dQ_\theta^{(n)}(\widetilde{Z}^{(n)})}{d\nu^{(n)}(\widetilde{Z}^{(n)})} = \prod_{m=1}^{n} \{k(Z_m)[1-(1-G(Z_m))(1-F(Z_m;\theta))]\}^{\Delta_m^{(0)}} \cdot$$

$$\cdot \{K(Z_m)(1-G(Z_m))f(Z_m;\theta)\}^{\Delta_m^{(1)}} \cdot \{K(Z_m)g(Z_m)(1-F(Z_m;\theta))\}^{\Delta_m^{(2)}}.$$

For $u \in R^1$ define $\theta_n = \theta_0 + \frac{u}{\sqrt{n}} \in \Theta$, where θ_0 is the true value of parameter θ and consider the logarithm of the likelihood ratio statistics (LRS)

$L_n(u) = \log\left\{dQ^{(n)}_{\theta_n}(\widetilde{Z}^{(n)})/dQ^{(n)}_{\theta_0}(\widetilde{Z}^{(n)})\right\}$. In the works of authors [1-3] under certain regularity conditions, is established for any $u \in R^1$ the following locally asymptotically normality type result for LRS:

$$L_n(u) = uJ^{1/2}(\theta_0)\zeta - \frac{u^2}{2}J(\theta_0) + R_n(u), \tag{1}$$

where $\zeta \stackrel{D}{=} N(0,1)$, $R_n(u) \stackrel{Q^{(n)}_{\theta_0}}{\rightarrow} 0$ for $n \to \infty$ and $J(\theta)$ - is a Fisher information corresponding to the observation $\widetilde{Z}_i$. We use (1) for investigation of estimators of θ.

Let $\{\pi(u), u \in \Theta\}$ is a non-negative measurable function and $l(d;\theta) = (d-\theta)^2$ is a loss function on the set $D \times \Theta$, where D - the set of possible estimates for θ. Consider the Bayesian - type estimator $\tilde{\theta}_n \in D$, defined as

$$\tilde{\theta}_n = arg \min_{d \in D} \frac{\int\limits_{\Theta} l(d;\theta)p_n(\widetilde{Z}^{(n)};\theta)\pi(\theta)d(\theta)}{\int\limits_{\Theta} p_n(Z^{(n)};\theta)\pi(\theta)d(\theta)}. \tag{2}$$

Note that if θ is r.v. with the a priori density π, then $\tilde{\theta}_n$ is Bayesian estimator for θ.

Theorem. *Let the following regularity conditions are hold:*

(I) The support $N_f = \{x : f(x;\theta) > 0\}$ *is independent on* θ*;*

(II) $f(x;\alpha) \neq f(x;\beta)$ *for* $\alpha \neq \beta$, $\alpha, \beta \in \Theta$;

(III) There are derivatives $\partial^i f(x;\theta)/\partial\theta^i$ *and* $\int\limits_{-\infty}^{\infty} |\partial^i f(x;\theta)/\partial\theta^i| dx < \infty$, $i = 1, 2$;

(IV) The function $\partial \log f(x;\theta_0)/\partial\theta$ *is of bounded variation;*

(V) $J(\theta_0) \neq 0$*;*

(VI) $\pi(\theta)$ *is continuous at the neighborhood of* θ_0 *and* $\pi(\theta_0) \neq 0$.

Then for $n \to \infty$, $\sqrt{n}(\tilde{\theta}_n - \theta_0) \Rightarrow N(0, (J(\theta_0))^{-1})$.

Note that the limit distribution of θ_n is independent of π.

References

1. A. A. Abdushukurov, N. S. Nurmuhamedova. Approximation of likelihood ratio statistics in competing risks model under random censoring from both sides. *Acta NUUz*, 2011. No. 4. P. 162-172.(In Russian).

2. A. A. Abdushukurov, N. S. Nurmuhamedova Approximation of likelihood ratio statistics in competing risks model. *Doclady Acad. Nauk. Uzbekistan*, 2012, No. 3.(In Russian).

3. B. A. Bobojonov, N. S. Nurmuhamedova. Approximation of likelihood ratio statistics in competing risks model under random censoring from the right. *In: Statistical Methods of Estimation and Hyphoteses testing.Perm State University.*, Perm. Issue 23, 2011, p. 152-165.(In Russian).

Stochastic analysis of traffic at non-regulated intersections

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We consider queueing systems with infinite number of servers and identical service times during a busy period. Service times on different busy periods are independent identically distributed random variables. For the process that defines the number of customers in the system the stationary distribution and the ergodicity condition are obtained. The distribution function of the system's busy period is found.

These results are applied to the analysis of traffic at non-regulated intersections. Assume that cars on a secondary road S_2 that intersects a major one-lane road S_1 can merge into the major road if there are no cars on S_1 on the certain distance I from the intersection. Similar models were analyzed in Tanner [1], Gideon and Pyke [2]. Such non-regulated intersections can be described by an $M|G|1$ queue with the unreliable server. $M|G|1$ systems with service interruptions were investigated in Gaver [3]. The results obtained there cannot be applied to our system directly because of the following assumptions we make which are specific to traffic models:

1. If a car on the secondary road reaches the intersection when the interval I on the main road is free and the queue is empty then the time required to merge into the main road for such a car is supposed to be zero ("skipping effect").

2. If a car appears in the interval I while another car on S_2 is passing the intersection then the car on S_2 stops and right after the interval I becomes free immediately crosses the main road (the residual time required for crossing is zero).

Using the results given in Afanasyeva [4] we obtain the necessary and sufficient ergodicity condition, limiting distribution of the number of customers in the system and investigate functioning of the system under heavy traffic assumptions.

Acknowledgements. The author is grateful to professor L.G.Afanasyeva for the statement of the problem and valuable discussions. This work was partially supported by RFBR grant N 10-01-00266a.

References

1. G. C. Tanner. The delay to pedestrians crossing a road. *Biometrika*, 1951,vol. 38, p. 383–392.

2. R. Gideon, R. Pyke. Markov renewal modelling of Poisson traffic at intersections having separate turn lanes. *Semi-Markov Models and Applications*, 1999, p. 285–310.

3. D. P. Gaver. A Waiting Line with Interrupted Service, Including Priorities. *J. Roy. Statist. Soc.*, 1962, vol. 24, p. 73–90.

4. L. G. Afanasyeva. Stochastic boundedness of cyclic queueing systems. *Journal of Mathematical Sciences*, 1992, vol. 59, p. 869–875.

A probability transformation with application to characteristic functions

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Definition 1. Let X be a random variable with the distribution function $F(x)$ and $\sigma^2 \equiv \mathsf{E}X^2 \in (0,\infty)$. The random variable X^* is said to have the X-*shape biased distribution*, if its distribution function $F^*(x)$ obeys the relation

$$dF^*(x) = \frac{x^2}{\sigma^2}dF(x), \quad x \in \mathbf{R},$$

or, which is the same, $F^*(x) = \sigma^{-2}\mathsf{E}X^2\mathbf{1}(X<x)$.

Definition 1 is a particular case of a more general definition given by Goldstein and Reinert [1], where, however, only the existence of this transformation was proved. Here we establish some properties of X^* and prove an exact estimate of the proximity of X^* to X in L_1-metric. It can be easily seen that:

1) $(X^*)^2$ has the same distribution as X^2-size biased distribution introduced by Goldstein and Rinott [2]: $(X^*)^2 \stackrel{d}{=} (X^2)^{(s)}$;

2) the symmetric binomial distribution is the fixed point of the shape biased transformation: $X^* \stackrel{d}{=} X$, if $\mathsf{P}(X=\pm\sigma) = 1/2$, $\sigma > 0$;

3) the characteristic functions of X^* and X are linked by the following relation:

$$f^*(t) \equiv \mathsf{E}e^{itX^*} = -\sigma^{-2}f''(t) = \frac{f''(t)}{f''(0)}, \quad f(t) = \mathsf{E}e^{itX}, \quad t \in \mathbf{R}.$$

Theorem 1. *For any random variable X with $\mathsf{E}X = 0$, $\mathsf{E}X^2 = 1$, and $\mathsf{E}|X|^3 < \infty$ the following inequality holds:*

$$L_1(X,X^*) \equiv \inf\{\mathsf{E}|\widetilde{X} - \widetilde{X}^*| \colon \widetilde{X} \stackrel{d}{=} X,\ \widetilde{X}^* \stackrel{d}{=} X^*\} \leqslant \mathsf{E}|X|^3$$

with the equality attained at any three-point distribution with an atom at zero.

The proof is based on the results of the works [3, 5].

Using theorem 1 and a result of [4] we obtain

Theorem 2. *Let X be an arbitrary random variable with $\mathsf{E}X = 0$, $\mathsf{E}X^2 = 1$, $\mathsf{E}|X|^3 < \infty$, and the characteristic function $f(t)$. Let X^* have an*

X-shape biased distribution with the characteristic function $f^*(t)$. Then for all $t \in \mathbf{R}$

$$|f(t) - f^*(t)| \equiv |f(t) + f''(t)| \leqslant$$
$$\leqslant 2\sin\left(\min\left\{L_1(X, X^*)\frac{t}{2}, \frac{\pi}{2}\right\}\right) \leqslant 2\sin\left(\min\left\{\frac{t\mathsf{E}|X|^3}{2}, \frac{\pi}{2}\right\}\right).$$

This research was supported by the Russian Foundation for Basic Research (projects 12-01-31125, 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a) and by the Ministry for Education and Science of Russia (project MK–2256.2012.1).

References

1. L. Goldstein, G. Reinert. Distributional transformations, orthogonal polynomials, and Stein characterizations. *Journal of Theoretical Probability*, 2005, vol. 18, No. 1, p. 237–260.

2. L. Goldstein, Y. Rinott. On multivariate normal approximations by Steins method and size bias couplings. *Journal of Applied Probability*, 1996, vol. 33, p. 1–17.

3. W. Hoeffding. The extrema of the expected value of a function of independent random variables. *Annals of Mathematical Statistics*, 1955, vol. 26, No. 2, p. 268–275.

4. V. Yu. Korolev, I. G. Shevtsova. An improvement of the Berry–Esseen inequality with applications to Poisson and mixed Poisson random sums. *Scandinavian Actuarial Journal*, 2012, No. 2, p. 81–105. Available online since 04 June 2010.

5. V. M. Zolotarev. Probability metrics. *Theory of Probability and its Applications*, 1983, vol. 28, No. 2, p. 264–287.

On the accuracy of the approximation of a characteristic function by the first terms of its Taylor expansion

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Let n be an integer, $n \geqslant 1$. For a random variable X with $\mathsf{E}|X|^n < \infty$ denote $\alpha_k = \mathsf{E}X^k$, $\beta_k = \mathsf{E}|X|^k$, $k = 1, 2, \ldots, n$. As is known,

$$\left|\mathsf{E}e^{itX} - \sum_{k=0}^{n-1}\frac{\alpha_k(it)^k}{k!}\right| \leqslant \frac{\beta_n|t|^n}{n!}, \quad t \in \mathbf{R}. \tag{1}$$

Moreover, it is known, that the factor $1/n!$ on the right-hand side of (1) cannot be made less. However, it does not mean that (1) cannot be sharpened. Indeed, H. Prawitz [1] showed that

$$\left|\mathsf{E}e^{itX}-\sum_{k=0}^{n-1}\frac{\alpha_k(it)^k}{k!}\right|\leqslant\frac{n|\alpha_n|+(n+2)\beta_n}{2(n+1)}\cdot\frac{|t|^n}{n!}\leqslant\frac{\beta_n|t|^n}{n!}.$$

Here we present an even sharper result.

Theorem 1. *For any random variable X such that $\mathsf{E}|X|^n<\infty$ with some integer $n\geqslant 1$, for all $t\in\mathbf{R}$ and $0\leqslant\lambda<1/2$ we have*

$$\left|\mathsf{E}e^{itX}-\sum_{k=0}^{n-1}\frac{\alpha_k(it)^k}{k!}-\frac{\lambda}{n!}\alpha_n(it)^n\right|\leqslant\frac{q_n(\lambda)}{n!}\beta_n|t|^n,\quad \textit{where}$$

$$q_n(\lambda)=n!\sup_{x>0}x^{-n}\left|e^{ix}-\sum_{k=0}^{n-1}\frac{(ix)^k}{k!}-\lambda\frac{(ix)^n}{n!}\right|,\quad 0\leqslant\lambda<\frac{1}{2}.$$

Corollary 1. *For any random variable X such that $\mathsf{E}|X|^n<\infty$ with some integer $n\geqslant 1$ for all $t\in\mathbf{R}$ we have*

$$\left|\mathsf{E}e^{itX}-\sum_{k=0}^{n-1}\frac{\alpha_k(it)^k}{k!}\right|\leqslant\inf_{0\leqslant\lambda<1/2}\left(\lambda\frac{|\alpha_n|}{\beta_n}+q_n(\lambda)\right)\frac{\beta_n|t|^n}{n!},$$

for $n=3$ the equality is attained for any $t\in\mathbf{R}$ at some symmetric tree-point distribution with an atom at zero.

In particular, $q_n(0)=1$, and, as it follows from [1],

$$q_n(\lambda)=\frac{n+2}{2(n+1)}\quad\text{for}\quad\lambda=\frac{n}{2(n+1)}.$$

Moreover, it can be made sure that $q_1(b)\geqslant q_1(0.311\ldots)=0.7246\ldots$, $q_2(b)\geqslant q_2(4/\pi^2)=2/\pi=0.6366\ldots$, $q_3(b)\geqslant q_3(0.446\ldots)=6\cdot 0.0991\ldots=0.5949\ldots$, and that the supremums in $q_n(\lambda)$ for $n=1,2,3$ are attained at the points $x=x_n(\lambda)$ such that $x_1(\lambda)=0$, if $\lambda\leqslant 1/4$; $x_2(\lambda)=0$, if $\lambda\leqslant 1/3$; $x_3(\lambda)=0$, if $\lambda\leqslant 3/8$; and, otherwise, $x_n(\lambda)$ are the unique roots of the equations

$$(2-\lambda x^2)\cos x+(1+\lambda)x\sin x-2=0,\quad n=1,$$

$$x(8-\lambda x^2)\sin x+4(\lambda x^2+x^2-4)\sin^2\frac{x}{2}-4x^2=0,\quad n=2,$$

$$2(\lambda x^4-18x^2+36)\cos x-6x(x^2(\lambda+1)-12)\sin x-(3-4\lambda)x^4-72=0,\ n=3,$$

in the interval $x\in(0,2\pi)$.

As it follows from the Jensen inequality, $|\alpha_n|/\beta_n\leqslant 1$, but in some particular cases, even sharper estimates can be obtained. For example, for $n=3$ the following statement holds.

Theorem 2. *For any $b \geqslant 1$ and any random variable X with $\mathsf{E}X = 0$, $\mathsf{E}X^2 = 1$, and $\mathsf{E}|X|^3 = b$ we have*

$$\mathsf{E}X^3 \leqslant A\,(b)\,\mathsf{E}|X|^3, \quad \textit{where } A(b) = \sqrt{\frac{1}{2}\sqrt{1+8b^{-2}} + \frac{1}{2} - 2b^{-2}},$$

with an inequality attained at the two-point distribution

$$\mathsf{P}\left(X = \frac{1}{2}\left(b \pm \sqrt{b^2+4}\right)\right) = \frac{2 + \frac{b}{2}\left(b \mp \sqrt{b^2+4}\right)}{b^2+4}.$$

Note that the function $A(b)$, $b \geqslant 1$, increases monotonically varying within the limits $0 = A(1) \leqslant A(b) < \lim\limits_{b\to\infty} A(b) = 1$.

Theorem 3. *For any random variable X with $\mathsf{E}X = 0$, $\mathsf{E}X^2 = 1$, $b \equiv \mathsf{E}|X|^3 < \infty$, and the characteristic function $f(t) = \mathsf{E}e^{itX}$, for all $t \in \mathbf{R}$ we have*

$$|\mathsf{E}\sin tX| \leqslant (1 + A(b))b|t|^3/12,$$
$$\left|f(t) - 1 + t^2/2\right| \leqslant b\gamma_3(b)t^3,$$
$$\left|f'(t) + t\right| \leqslant b\gamma_2(b)t^2,$$
$$\left|f''(t) + 1\right| \leqslant b\gamma_1(b)t,$$
$$|f''(t) + f(t)| \leqslant 2\sin\left(\min\{bt, \pi\}/2\right),$$

where

$$\gamma_k(b) = \frac{1}{k!}\inf_{0\leqslant\lambda<1/2}(\lambda A(b) + q_k(\lambda)), \quad k = 1, 2, 3,$$

the function $A(b)$ being defined in Theorem 2. If $\mathsf{E}X^3 = 0$, then one can assign $A(b) \equiv 0$.

This research was supported by the Russian Foundation for Basic Research (projects 12-01-31125, 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a) and by the Ministry for Education and Science of Russia (project MK–2256.2012.1).

References

1. H. Prawitz. Noch einige Ungleichungen für Charakteristische Funktionen. *Scandinavian Actuarial Journal*, 1991, No. 1, p. 49–73.

On the absolute constants in the Berry–Esseen-type inequalities

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Let $X_1, \ldots, X_n$ be independent random variables such that

$$\mathsf{E}X_j = 0, \quad \mathsf{E}X_j^2 = \sigma_j^2, \quad \mathsf{E}|X_j|^3 = \beta_{3,\,j} < \infty, \quad j = 1, 2, \ldots, n, \quad \sum_{j=1}^{n} \sigma_j^2 = 1.$$

Denote

$$\ell_n = \sum_{j=1}^{n} \beta_{3,\,j}, \quad \tau_n = \sum_{j=1}^{n} \sigma_j^3,$$

$$\Delta_n = \sup_x \big|\mathsf{P}(X_1 + \ldots + X_n < x) - \Phi(x)\big|, \quad n = 1, 2, \ldots,$$

$\Phi(x)$ being the standard normal distribution function. It is easy to verify that under the above assumptions for any $n \geqslant 1$ we have $\ell_n \geqslant \tau_n \geqslant n^{-1/2}$. Using new estimates for characteristic functions presented in [7, 8], as well as the asymptotic estimates from [4, 5, 6] we prove that under the above conditions for all $n \geqslant 1$ the following inequalities hold.
In the general case:

$$\begin{aligned}
\Delta_n &\leqslant 0.39885(\ell_n + 0.4\tau_n) \leqslant 0.5584\ell_n, \\
\Delta_n &\leqslant 0.36266(\ell_n + 0.54\tau_n) \leqslant 0.5585\ell_n, \\
\Delta_n &\leqslant 0.3129(\ell_n + 0.922\tau_n);
\end{aligned}$$

in the i.i.d. case:

$$\begin{aligned}
\Delta_n &\leqslant 0.4693\ell_n, \\
\Delta_n &\leqslant 0.3322(\ell_n + 0.429\tau_n) \leqslant 0.3355(\ell_n + 0.415\tau_n) \leqslant 0.4748\ell_n, \\
\Delta_n &\leqslant 0.3031(\ell_n + 0.646\tau_n) \leqslant 0.3351(\ell_n + 0.489\tau_n);
\end{aligned}$$

in the non-i.i.d. case for symmetrically distributed summands:

$$\begin{aligned}
\Delta_n &\leqslant 0.5582\ell_n, \\
\Delta_n &\leqslant 0.3425(\ell_n + 0.63\tau_n) \leqslant 0.5583\ell_n, \\
\Delta_n &\leqslant 0.2895(\ell_n + \tau_n) \leqslant 0.5584\ell_n;
\end{aligned}$$

in the i.i.d. case for symmetrically distributed summands:

$$\begin{aligned}
\Delta_n &\leqslant 0.29489(\ell_n + 0.587\tau_n) \leqslant 0.4680\ell_n, \\
\Delta_n &\leqslant 0.2730(\ell_n + 0.732\tau_n).
\end{aligned}$$

These inequalities improve those obtained in [1, 2, 3, 9].

This research was supported by the Russian Foundation for Basic Research (projects 12-07-33063, 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a) and by the Ministry for Education and Science of Russia (project MK–2256.2012.1).

References

1. V. Yu. Korolev, I. G. Shevtsova. An improvement of the Berry–Esseen inequality with applications to Poisson and mixed Poisson random sums. *Scandinavian Actuarial Journal*, 2012, No. 2, p. 81–105. Available online since 04 June 2010.

2. V. Yu. Korolev, I. G. Shevtsova. A new moment-type estimate of convergence rate in the Lyapunov theorem. *Theory Probability and its Applications*, 2011, vol. 55, No. 3, p. 505–509.

3. I. G. Shevtsova. On the absolute constants in the Berry-Esseen type inequalities for identically distributed summands. *arXiv: 1111.6554v1*, 28 November, 2011.

4. I. G. Shevtsova. On the accuracy of the normal approximation for sums of independent random variables. *Doklady Mathematics*, 2012, vol. 85, No. 2, p. 274–278.

5. I. G. Shevtsova. On the accuracy of the normal approximation for sums of independent symmetric random variables. *Doklady Mathematics*, 2012, vol. 85, No. 2, p. 292–296.

6. I. G. Shevtsova. Moment-type estimates with asymptotically optimal structure for the accuracy of the normal approximation. *Annales Mathematicae et Informaticae*, 2012, vol. 39, p. 241–307.

7. I. G. Shevtsova. On the accuracy of the approximation of a characteristic function by the first terms of its Taylor expansion. *Abstracts of the XXX Seminar on Stability Problems for Stochastic Models*, 2012.

8. I. G. Shevtsova. A probability transformation with application to characteristic functions. *Abstracts of the XXX Seminar on Stability Problems for Stochastic Models*, 2012.

9. I. S. Tyurin. On the convergence rate in Lyapunov's theorem. *Theory Probability and its Applications*, 2011, vol. 56, No. 4, p. 808–811.

Structural improvements of non-uniform convergence rate estimates in the central limit theorem for sums of independent random variables

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Let $X_1, \ldots, X_n$ be independent random variables such that

$$\mathsf{E}X_j = 0, \quad \mathsf{E}X_j^2 = \sigma_j^2, \quad \mathsf{E}|X_j|^3 = \beta_{3,\,j} < \infty, \quad j = 1, 2, \ldots, n, \quad \sum_{j=1}^{n} \sigma_j^2 = 1.$$

Denote

$$\ell_n = \sum_{j=1}^{n} \beta_{3,\,j}, \quad \tau_n = \sum_{j=1}^{n} \sigma_j^3,$$

$$\Delta_n(x) = \big|\mathsf{P}(X_1 + \ldots + X_n < x) - \Phi(x)\big|, \quad n = 1, 2, \ldots,$$

$\Phi(x)$ being the standard normal distribution function. It is easy to verify that under the above assumptions for any $n \geqslant 1$ we have $\ell_n \geqslant \tau_n \geqslant n^{-1/2}$. Using new uniform estimates presented in [5], we prove that under the above conditions the following inequalities hold for all $n \geqslant 1$:

$$\sup_{x\in\mathbf{R}} |x|^3 \Delta_n(x) \leqslant \begin{cases} 21.31\ell_n, \\ 19.46(\ell_n + 0.4\tau_n), \\ 17.95(\ell_n + 0.922\tau_n), \end{cases} \tag{1}$$

$$\sup_{x\in\mathbf{R}} \left(1 + |x|^3\right) \Delta_n(x) \leqslant \begin{cases} 21.87\ell_n, \\ 19.85(\ell_n + 0.4\tau_n), \\ 18.27(\ell_n + 0.922\tau_n) \end{cases} \tag{2}$$

in the general case, and

$$\sup_{x\in\mathbf{R}} |x|^3 \Delta_n(x) \leqslant \begin{cases} 16.90\ell_n, \\ 15.62(\ell_n + 0.429\tau_n), \\ 15.36(\ell_n + 0.646\tau_n), \end{cases} \tag{3}$$

$$\sup_{x\in\mathbf{R}} \left(1 + |x|^3\right) \Delta_n(x) \leqslant \begin{cases} 17.37\ell_n, \\ 15.96(\ell_n + 0.429\tau_n), \\ 15.66(\ell_n + 0.646\tau_n) \end{cases} \tag{4}$$

in the i.i.d. case. Moreover, the most exact estimate among the three estimates for each of the four quantities under consideration is given by the first inequality, if $b \equiv \ell_n/\tau_n \leqslant b_1$, by the second inequality — if $b_1 \leqslant b \leqslant b_2$, and by the third one — if $b \geqslant b_2$, where approximately

$$\begin{array}{lll} b_1 = 4.2, & b_2 = 5.8, & \text{in (1)}, \\ b_1 = 3.9, & b_2 = 5.6, & \text{in (2)}, \\ b_1 = 5.2, & b_2 = 12.2, & \text{in (3)}, \\ b_1 = 4.8, & b_2 = 11.2, & \text{in (4)}. \end{array}$$

The presented results improve those obtained in [2, 3, 4].

Furthermore, it is shown that the absolute constant C in the estimates like

$$\sup_{x\in\mathbf{R}} |x|^3\Delta_n(x) \leqslant C\ell_n,$$

can be replaced by a non-increasing function $C(x)$, given in the explicit form, such that for all $n \geqslant 1$ and $x \geqslant 0$

$$\sup_{|t|\geqslant x} |t|^3\Delta_n(t) \leqslant C(x)\ell_n.$$

Moreover, the function $C(x)$ is optimal in the asymptotical sense, i.e. it's limit value

$$\lim_{x\to\infty} C(x) = 1$$

can not be made less (see [1]). Some particular values of $C(x)$ are presented below:

non-i.i.d.case: $C(4) \leqslant 17.05$, $C(7) \leqslant 7.53$, $C(10) \leqslant 4.65$, $C(30) \leqslant 1.79$,

i.i.d.case: $C(4) \leqslant 14.50$, $C(7) \leqslant 7.47$, $C(10) \leqslant 4.64$, $C(30) \leqslant 1.79$.

This research was supported by the Russian Foundation for Basic Research (projects 12-07-33063, 11-01-12026-ofi-m, 11-01-00515a and 11-07-00112a) and by the Ministry for Education and Science of Russia (project MK–2256.2012.1).

References

1. G. P. Chistyakov. On a problem of A. N. Kolmogorov. *Journal of Mathematical Sciences*, 1994, vol. 68, No. 4, p. 604–625.

2. M. E. Grigorieva, S. V. Popov. On non-uniform convergence rate estimates in the central limit theorem. *Systems and Means of Informatics*, 2012, vol. 22, No. 1, p. 180–204 (in Russian).

3. Yu. S. Nefedova, I. G. Shevtsova. Structural improvements of nonuniform convergence rate estimates in the central limit theorem with applications to Poisson random sums. *Doklady Mathematics*, 2011, vol. 84, No. 2, p. 675–680.

4. Yu. S. Nefedova, I. G. Shevtsova. On non-uniform convergence rate estimates in the central limit theorem. *Theory Probability and its Applications*, 2012, vol. 57, No. 1, p. 62–97 (in Russian).

5. I. G. Shevtsova. On the absolute constants in the Berry–Esseen-type inequalities. *Abstracts of the XXX Seminar on Stability Problems for Stochastic Models*, 2012.

Investigation of harmonics of Fourier spectra of non-Gaussian processes of structural plasma turbulence

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Spectral analysis is one of the most powerful tools for experimental data processing in different fields including investigation of plasma turbulence. To separate the parameters of the spectrum obtained by spectrometers, spectrograph or by the estimating of ADC's (Analog-to-digital converter) sample, spectrum should be decomposed into the components. The problem is ill-posed because of incomplete data. Moreover, it has the unique solution only under additional assumptions about the fine structure of the modelled object [1, 2]. Also it is impossible to obtain important spectral information about the functioning of plasma turbulence by the traditional approach implying spectrum's approximation by Kolmogorov-Obukhov model or shot (fluctuation) noise model.

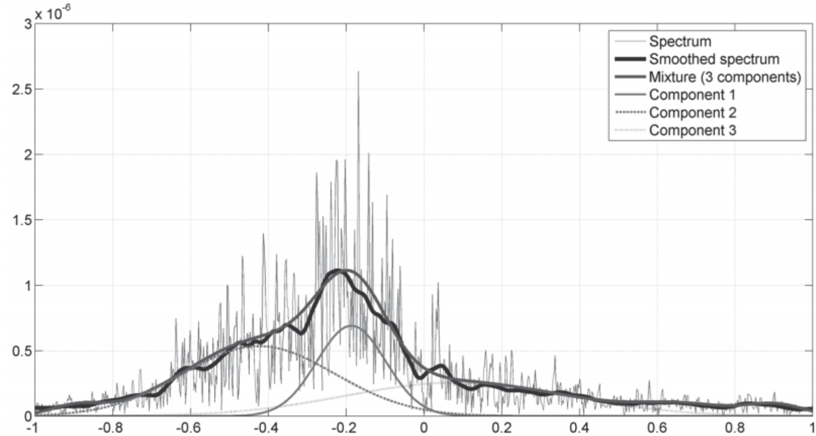

Figure 1: Spectrum decomposition.

In the last decade strong structural state of the low-frequency turbulence were found during studies of low-frequency (100 MHz) plasma fluctuations [3]. This state appears in a stationary plasma in an open thermodynamic system with a constant inflow and outflow of energy as a result of transient processes: the growth and saturation of instabilities and appearance of stochastic plasma structures. The turbulence is successfully described by a mathematical model based on compound Cox process [4].

By analyzing of the increments of plasma fluctuations, the number of processes, which form the initial ion-acoustic turbulence, has been revealed [5]. The next step of investigation should be based on analysis of the spectra, since it allows to identify the type of instability, mechanism of the functioning of turbulence, the proportion of ion-acoustic solitons and drift vortices.

The idea and methodology of such analysis are suggested in [6]. Example of spectrum's decomposition into the components is shown in fig. 1.

Acknowledgements. The work is supported by the Russian Foundation for Basic Research, project 12-07-31267, and the Ministry for Science and Education of Russian Federation, state contract P770.

References

1. K. Seshadri, R. Jones. The shapes and intensities of infrared absorption bands. *Physics-Uspekhi*, 1965, vol. 85, No. 1, p. 87–145.

2. S. A. Akhmanov, S. Yu. Nikitin. *Physical Optics*, 2004, Moscow: Nauka.

3. V. Yu. Korolev and N. N. Skvortsova (Eds). *Stochastic Models of Structural Plasma Turbulence*, 2006, Untrecht: VSP.

4. V. Yu. Korolev. *Probabilistic-statistical methods for the decomposition of the volatility of chaotic processes*, 2011, Moscow: Moscow University Press.

5. G. M. Batanov, A. K. Gorshenin, V. Yu. Korolev, D. V. Malakhov, N. N. Skvortsova. The evolution of probability characteristics of low-frequency plasma turbulence. *Mathematical Models and Computer Simulations*, 2012, vol. 4. No. 1, p. 10–25.

6. A. K. Gorshenin, V. Yu. Korolev, D. V. Malakhov, N. N. Skvortsova. Analysis of fine stochastic structure of chaotic processes by kernel estimators. *Mathematical Modelling*, 2011, vol. 23, No. 4, p. 83–89.

Recurrent sequence of parallel-parallel-serial connections

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In the reliability theory parallel-serial connections play important role. These connections are widely used in electrotechnics, in computer networks etc. A specific of these connections is a possibility to calculate their reliability by algorithms with linear complexity by a number of arcs.

Characteristics of networks sparseness arouse large interest last years. Network sparseness means that powers of nodes (a number of incident arcs) is bounded by some positive number (see Raigorodsky [1] and large bibliography in this article). Simultaneously a distribution of numbers of connectivity components in different random networks are analyzed intensively now Timashev [2].

Stochastic modeling and statistical processing of internet type networks data showed that nodes powers have distribution with heavy tails. This circumstance makes actual to consider parallel-serial connections which are free of this lack. In this paper numbers of connectivity components in recurrent sequence of connections obtained by parallel or serial linking of new arc are considered. For this sequence central limit theorem is proved and parameters of limit normal distribution are calculated.

A problem to calculate a mean and mainly a variance of limit normal distribution in this model is technically sufficiently complicated. Convenient algorithm of such symbolic calculations based on a combination of central limit theorem for discrete Markov chains Romanovskiy [3] and some relations for conditional means of numbers of connectivity components in parallel-serial connections is constructed.

Consider the sequence $\mathcal{A}_n,\ n \geq 1$, of ports defined recursively by serial or parallel connection of new arc b_n to the port $\mathcal{A}_n$. Denote a type of connection by $||$ or $\to$, accordingly. Suppose that random variable ω_n characterizes a type of the arc b_n connection to the port $\mathcal{A}_n$ and put

$$\pi_{\to} = P(\omega_n = \to),\ \pi_{||} = P(\omega_n = ||) = 1 - \pi_{\to},\ 0 < \pi_{\to} < 1.$$

Here random variable β_n characterizes a state of the arc b_n :

$$P(\beta_n = 1) = P(b_n \text{ in working state}) = p,\ P(\beta_n = 0) = 1 - p = q,\ 0 < p < 1.$$

The sequences of random variables $\{\omega_n,\ n \geqslant 1\}$, $\{\beta_n,\ n \geqslant 1\}$ are independent and each of them consists of independent and identically distributed random variables.

The port $\mathcal{A}_n$ with randomly working arcs is characterized by random vector (α_n, η_n), there α_n is an indicator of a connectivity between initial and final nodes of parallel-sequential connection $\mathcal{A}_n$ and η_n is a number of connectivity components in $\mathcal{A}_n$. Then for any real t

$$P\left(\frac{\eta_n - nA}{\sqrt{Bn}} > t\right) \to P(N(0,1) > t),\ n \to \infty$$

where $N(0,1)$ is standard normal random variable and

$$A = Q\pi_{\to}q,\ B = \pi_{\to}qQ(1 - \pi_{\to}qQ + 2PQ).$$

The parameters A, B calculation is based on the following equalities. Denote

$$A_n = M\left(\eta_n | a_n = 1\right),\ B_n = M\left(\eta_n | a_n = 0\right),\ P = \frac{\pi_{\|}p}{\pi_{\|}p + \pi_{\to}q},\ Q = 1 - P$$

$$A'_n = M\left(\eta_n^2 | a_n = 1\right),\ B'_n = M\left(\eta_n^2 | a_n = 0\right)$$

then

$$A_{n+1} = \frac{1}{P}\left(A_nP\pi_{\to}p + A_nP\pi_{\|}p + (B_n - 1)Q\pi_{\|}p + A_nP\pi_{\|}q\right),$$

$$B_{n+1} = \frac{1}{Q}\left(B_nQ\pi_{\to}p + (A_n + 1)P\pi_{\to}q + (B_n + 1)Q\pi_{\to}q + B_nQ\pi_{\|}q\right),$$

$$A'_{n+1} = \frac{1}{P}\left(A'_nP\pi_{\to}p + A'_nP\pi_{\|}p + (B'_n - 2B_n + 1)Q\pi_{\|}p + A'_nP\pi_{\|}q\right),$$

$$B'_{n+1} = \frac{1}{Q}\left(B'_nQ\pi_{\to}p + (A'_n + 2A_n + 1)P\pi_{\to}q + (B'_n + 2B_n + 1)Q\pi_{\to}q + \right.$$

$$\left. + B'_nQ\pi_{\|}q\right).$$

Acknowledgements This paper is supported by RFBR grant 12-01-00114-a and FEB RAS grant 12-III-A-01I-002.

References

1. A. M. Raigorodskiy. Models of random graphs and their application, *Proceedings of MPhTI*, 2010, vol. 2, No. 4, p. 130–140.

2. A. Timashev. Asymptotic expansions for distributions of numbers of connectivity components in random images and decompositions, *Discrete mathematics*, 2011, vol. 23, No. 2, p. 66–75.

3. V. Romanovskiy. *Discrete Markov chains*, 1949, State Publishing House of Technical-Theoretical Literature, Moscow-Leningrad.

Some characterizations of symmetry

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Braverman [1] obtained the following characterization of symmetry about the origin. Let X and Y be independent and identically distributed random variables, and let $0 < p < 2$. Then X is symmetric about 0 if and only if

$$\mathrm{E}|X - Y|^p = \mathrm{E}|X + Y|^p.$$

Some extentions of this result were obtained in Ushakov [2] and Ushakov [3]. In this work, we present some characterizations (of similar form) of general symmetry (symmetry about an arbitrary point).

Denote a median of the random variable X by $m(X)$. If X and Y have the same distribution, we denote this by $X \stackrel{d}{=} Y$. The random variable X is called to be symmetric if there exists a real number c such that

$$X - c \stackrel{d}{=} -(X - c).$$

Let X and Y be independent and identically distributed random variables.

Theorem 1. *Let $0 < p < 2$, and $\mathrm{E}|X|^p < \infty$. Then X is symmetric if and only if*

$$\mathrm{E}|X - Y|^p = \mathrm{E}|X + Y - 2\mathrm{m}(X)|^p. \tag{1}$$

If the expectation of X exists, then we can formulate the characterization in terms of the expectation rather than in terms of the median.

Theorem 2. *Let $0 < p < 2$, and $\mathrm{E}|X|^{\max\{1,p\}} < \infty$. Then X is symmetric if and only if*

$$\mathrm{E}|X - Y|^p = \mathrm{E}|X + Y - 2\mathrm{E}X|^p. \tag{2}$$

Consider now moments of order $2 < p < 4$. It turns out that a characterization analogous to Theorem 2 still holds in this case but a characterization analogous to Theorem 1 does not.

Theorem 3. *Let $2 < p < 4$, and $\mathrm{E}|X|^p < \infty$. Then X is symmetric if and only if*

$$\mathrm{E}|X - Y|^p = \mathrm{E}|X + Y - 2\mathrm{E}X|^p.$$

In contrast to the case $0 < p < 2$, where characterizations both in terms of the median and in terms of the expectation hold, in the case $2 < p < 4$ the characterization in terms of the median does not hold. This follows from the following

Proposition 1. *For any $2 < p < 4$ there exist independent and identically distributed random variables X and Y which are nonsymmetric but*

$$\mathrm{E}|X - Y|^p = \mathrm{E}|X + Y - 2m(X)|^p.$$

In the case $p \geqslant 4$ neither characterization (1) nor characterization (2) holds. This follows from

Proposition 2. *For any $p \geqslant 4$, there exist*

(a) nonsymmetric independent and identically distributed random variables X and Y such that $m(X) = 0$, and $\mathrm{E}|X - Y|^p = \mathrm{E}|X + Y|^p$,

(b) nonsymmetric independent and identically distributed random variables X and Y such that $\mathrm{E}X = 0$, and $\mathrm{E}|X - Y|^p = \mathrm{E}|X + Y|^p$.

References

1. M. Sh. Braverman. Characteristic properties of normal and stable distributions. *Theory of Probability and its Applications*, 1985, vol. 30, p. 440-448.

2. N. G. Ushakov. One characterization of symmetry. *Statistics and Probability Letters*, 2011, vol. 81, p. 614-617.

3. N. G. Ushakov. Some inequalities for absolute moments. *Statistics and Probability Letters*, 2011, vol. 81, p. 2011-2015.

Multivariate geometric random sums and their asymptotic distributions

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In this report, we use multivariate geometric distribution to generalize the notion of geometric random sum to the multidimensional case. To date have been studied limit distributions, which approximate the geometric sum in the form

$$\sum_{j=1}^{M} X^{(j)},$$

where $X^{(j)} = \left(X_1^{(j)}, \dots, X_k^{(j)}\right)$ are independent k-dimensional random vectors, M is a random variable with geometric distribution; M and $X^{(j)}$ $(j = 1, 2, \dots)$ are independent. Note that the number of terms will be the same for each component.

In this paper we consider the more general case. The number of random variables M_j $(j = 1, \dots, k)$ will be different for each component, while values of M_j will be independent.

Let $\mathcal{E} = \{\epsilon\}$ is a set of $k-$dimensional indices; $\epsilon = (\varepsilon_1, \dots, \varepsilon_k)$ and each component of ε_i is 0 or 1; $\mathcal{E}_\nu$ is a set of $k-$dimensional indices for which $\varepsilon_\nu = 1$.

N_ε are independent geometrically distributed random variables:

$$P(N_\varepsilon = k) = p_\varepsilon q_\varepsilon^{k-1}, \quad k = 1,\, 2,\, \ldots \quad q_\varepsilon = 1 - p_\varepsilon.$$

Let $N_\varepsilon = \infty$ with $p_\varepsilon = 0$.
Let

$$M_\nu = \min_{\varepsilon \,\in\, \mathcal{E}_\nu} \{N_\varepsilon\}\,, \quad \nu = 1,\, \ldots,\, k$$

The distribution of the vector $M = (M_1,\, \ldots,\, M_k)$ is introduced and studied by the authors (currently in print) and is called *multivariate geometric distribution* (**MVG**). Multivariate geometric distribution has properties similar to those of one-dimensional geometrical laws.

Multivariate geometric random sum is called a random vector sum of the form

$$Z = (Z_1, \ldots, Z_k) = \left(\sum_{j=1}^{M_1} X_1^{(j)}, \ldots, \sum_{j=1}^{M_k} X_k^{(j)} \right),$$

where M_ν are defined above, $X_\nu^{(j)}$ $(\nu = 1, \ldots, k)$ are independent random variables identically distributed for each ν with the known characteristic function

$$E\, e^{i\, t_\nu\, X_\nu} = \phi_\nu(t_\nu),$$

and values M_ν and $X_\nu^{(j)}$ are independent.

Multivariate geometric random sums include the two extreme cases.

With $M_\nu = N_1$ $(\nu = 1, \ldots, k)$, $\quad N_\varepsilon = 0,\ \varepsilon \neq 1$, we obtain the standard geometric sums.

With $M_\nu = N_{\varepsilon_\nu}$, where $\varepsilon_\nu = (0,\, \ldots,\, 0,\, \underset{\nu}{1},\, 0,\, \ldots,\, 0)$, $\quad N_\varepsilon = 0,\ \varepsilon \neq \mathcal{E}_\nu, (\nu = 1, \ldots, k)$, each component will be a univariate geometric random sum while components themselves are independent.

The characteristic functions of multivariate random sums are found as well as their projections on an arbitrary coordinate hyperplane. The sufficient conditions for weak convergence of these sums to the multivariate exponential distribution and to the generalized multivariate Laplace distribution are given.

It is shown that the limit distributions of Z by the corresponding normalization can be:
— multivariate exponential distribution introduced by Marshall and Olkin (A multivariate exponential distribution. Y. Amer. Statist. Assoc., 1967, 62, 30-34);
— multivariate generalized Laplace distribution introduced earlier by the authors (Zolotukhin I.V., Zolotukhina L.A. New Class of Multivariate generalized Laplace Distributions. XXIV International Seminar on Stability Problems for Stochastic Models).

ASSOCIATED WORKSHOP
VI International Workshop "Applied Problems in Theory of Probabilities and Mathematical Statistics Related to Modeling of Information Systems"
Summer Session

Minimax estimation in regression under sum polygon generated constraint

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Minimax approach gives one of the prospective tools to solve the estimation problems in the regression under the observation model uncertainty. Usually, the uncertainty set is determined by available *prior* information, and described by the geometric constraints to the uncertain factors [4] and/or statistical constraints to their distribution [1,2]. In [3] there was an attempt to measure a relevance between the uncertainty and realized observations, and construct an additional uncertainty constraint. The considered relevance index was based on the likelihood function.

The aim of this paper is to solve the minimax estimation problem under the relevance constraint based on the sum polygon of the available observations.

Let us consider the following observation model:

$$Y_n = A(\gamma) + B(\gamma)V_n, \quad n = \overline{1, N}. \tag{1}$$

Here $\gamma \in \mathcal{C} \subseteq \mathcal{B}(\mathbb{R}^m)$ is an unobservable estimated vector ($\mathcal{C}$ is a compact set), $Y \triangleq \{Y_n\}_{n=\overline{1,N}}$ is a vector of observations and $V \triangleq \{V_n\}_{n=\overline{1,N}}$ is a random vector of i.i.d. centered normalized observation errors with the known pdf $\varphi_V(v)$. The vector γ is supposed to be random with unknown distribution F, belonging to the set $\mathbb{F}_K$ of admissible distributions described below.

The model (1) is defined on the family of the canonical spaces $\{(\Omega, \mathcal{F}, \mathbf{P}_F)\}_{F\in\mathbb{F}}$, where $\Omega \triangleq \mathcal{C} \times \mathbb{R}^N$, $\mathcal{F} \triangleq \mathcal{B}(\mathcal{C} \times \mathbb{R}^N)$, $\mathbf{P}_F\{\gamma \in dq,\ V_1 \in dv_1,\ \dots,\ V_N \in dv_N\} \triangleq F(dq)\prod_{n=1}^N \varphi_V(v_n)dv_n$.

Given the value γ the observations Y can be considered as i.i.d. random values, which pdf is equal to $\varphi_V(v)$ after some shifting and scaling. The sum polygon of this sample has the form $\overline{P}_N(y, Y) \triangleq \frac{1}{N}\sum_{t=1}^N \mathbf{I}(y - Y_n)$. On the other hand, the cdf of any observation Y_n for a fixed distribution F can be calculated as $P(y, F) \triangleq \int_{-\infty}^{y} \varphi_V(\frac{u-A(q)}{B(q)})F(du)$.

The relevance index based on the sum polygon is the following value

$$\mathfrak{K}(Y, F) \triangleq \|\overline{P}_N - P\|_\infty = \sup_{y \in \mathbb{R}} |\overline{P}_N(y, Y) - P(y, F)|. \tag{2}$$

The index presents the well-known Kolmogorov statistic (distance) used in the goodness-of-fit test.

Let $\mathbb{F}$ be a set of all probability distributions with the support lying in $\mathcal{C}$. We suppose the set $\mathbb{F}_K$ of admissible distributions is some nonempty convex $*$-weakly compact subset of $\mathbb{F}$ with an additional *constraint generated by the sum polygon of the level* K:

$$\mathfrak{K}(Y, F) \leqslant K \tag{3}$$

for all $F \in \mathbb{F}_K$ and some fixed level $K > 0$.

The estimated signal is a known continuous function $h(\gamma) : \mathcal{C} \times \mathbb{R}^m \to \mathbb{R}^\ell$ of γ, such that $\max_{q \in \mathcal{C}} \|h(q)\| < \infty$, and the set $\mathbb{H}$ of admissible estimates consists of all functions $\overline{h}(Y) : \mathbb{R}^N \to \mathbb{R}^l$, such that $\sup_{F \in \mathbb{F}} \mathbf{E}_F \left\{\|\overline{h}(Y)\|^2\right\} < \infty$.

The loss function is a conditional mean square estimation error

$$J(\overline{h}, F|y) \triangleq \mathbf{E}_F \left\{\|h(\gamma, X) - \overline{h}(Y)\|^2 | Y = y\right\}, \tag{4}$$

and the guaranteeing estimation criterion

$$J^*(\overline{h}|y) \triangleq \sup_{F \in \mathbb{F}_K} J(\overline{h}, F|y) \tag{5}$$

characterizes the maximal loss for a fixed estimator $\overline{h}$ and observations $Y = y$.

The minimax estimation problem for the vector h is to find an estimator $\widehat{h}(\cdot)$, such that

$$\widehat{h}(y) \in \operatorname*{Argmin}_{\overline{h} \in \mathbb{H}} J^*(\overline{h}|y). \tag{6}$$

Using the notation $\widehat{g}^F(y) \triangleq \mathbf{E}_F\{g(\gamma)|Y = y\}$ for the conditional expectation, we introduce the dual criterion

$$J_*(F|y) \triangleq \widehat{\|h\|^2}^F(y) - \|\widehat{h}^F(y)\|^2 \tag{7}$$

and the dual optimization problem

$$\widehat{F}(y) \in \operatorname*{Argmax}_{F \in \mathbb{F}_K} J_*(F|y). \tag{8}$$

Theorem 1. The loss function $J(\overline{h}, F|y)$ has a saddle point $(\widehat{F}(y), \widehat{h}(y))$ on the set $\mathbb{F}_K \times \mathbb{H}$: the least favorable distribution (LFD) $\widehat{F}(y)$ is a solution to a dual problem (8), and $\widehat{h}(y) = \widehat{h(\gamma)}^{\widehat{F}}(y)$ is a conditional expectation of $h(\gamma)$ given the observation $Y = y$ calculated under the LFD $\widehat{F}(y)$.

The $\widehat{h}(y)$ provides a solution to (6); it is invariant w.r.t. the LFD's choice: if $\widehat{F}'(y)$ and $\widehat{F}''(y)$ are two different LFD, then $\widehat{h(\gamma)}^{\widehat{F}'}(y) = \widehat{h(\gamma)}^{\widehat{F}''}(y)$.

There exists a variant of the LFD $\widehat{F}(y)$ concentrated at most at $\ell+2$ points of the set $\mathcal{C}$.

References

1. J. Berger. *Statistical Decision Theory and Bayesian Analysis*, Springer, 1985, New York.

2. D. Blackwell and M. Girshick. *Theory of Games and Statistical Decisions*, Wiley, 1954, New York.

3. A. Borisov. Posterior Minimax Estimation Under Likelihood Constraints. *Autom. Rem. Contr.*, 2012, to appear.

4. G. Di Masi, A. Gombiani and A. Kurzhansky. *Modeling, Estimation and Control of Systems with Uncertainty*, Birkhauser, 2007, München.

Effective bandwidth estimation in fluid system with regenerative input

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The Effective Bandwidth (EB) estimation attracts an increasing attention nowadays. The EB is a powerful metric which is used in admission control to satisfy QoS requirements in communication networks, in particular concerning the loss probability and packet delay [1].

We consider a buffered fluid queue with a positive recurrent regenerative input and an unknown constant service rate C. The EB problem is to find a minimal rate C which ensures a given value Γ of the overflow/loss probability. Based on large deviation theory [1, 2] we then use an exponential approximation

$$P(W > b) \asymp e^{-\theta^* b},\ b \to \infty,$$

where W is stationary workload, to find unknown exponent θ^* of the approximation as $\theta^* = -\ln \Gamma / b$. (Notation $\asymp$ stands for the logarithmic asymptotics.) Finally, the scaled limiting cumulant generating function $\Lambda(\theta^*)$ of the input process is calculated to obtain required rate C as

$$C = \frac{\Lambda(\theta^*)}{\theta^*}.$$

With the exception of the simplest cases (for instance, Poisson input), an analytical form of function Λ is not available, and simulation is used in order to determine C.

In this work we establish the strong consistency of a new regenerative EB estimator $\hat{\Lambda}_n(\theta^*)$ based on n observations of the input (and under some moment assumptions excluding heavy-tailed distributions). Namely, we show that with probability 1,

$$\hat{\Lambda}_n(\theta^*) \to \frac{\ln Ee^{\theta^* X}}{E\alpha},\ n \to \infty,$$

where X is the amount of the workload arrived during regeneration cycle and α is the regeneration period. (Earlier this result as a lower bound has been established in [4].) Note that $E\alpha < \infty$ by positive recurrence. The key element of the analysis is the *strong invariance principle* for renewal process formed by regenerations of the input [3]. Simulations illustrate properties of the estimator for various regenerative inputs.

The work is done in the framework of the Strategic development program of Petrozavodsk State University, and also supported by RFBR, project 10-07-00017.

References

1. A. Ganesh, N. O'Connell and D. Wischik. *Big Queues*, 2004, Springer-Verlag, Berlin.

2. P. W. Glynn, W. Whitt. *Logarithmic asymptotics for steady-state tail probabilities in a single-server queue*, JAP, 31, 1994, pp. 131-156.

3. M. Csörgő, L. Horvath and J. Steinebach. *Invariance principles for renewal processes*, 1987, The Annals of Probability, v.15, No. 4, pp. 1441-1460.

4. A.Borodina, I. Dyudenko, E. Morozov. *Speed-up estimation of overflow probability in regenerative queueing systems*, Journal of Applied and Industrial Mathematics, Vol. 16, No. 4, 577-593, 2009 (in Russian).

Performance analysis of a holographic Walsh-Hadamard Transformation based binary files encoding

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Walsh-Hadamard tramsformation (WHT) (and WHT-based codes) of digital random sequences is used widely in many computer science and data transmission areas, for example, for image data transmission. Recently we proposed (S.Dolev and S.Frenkel (2010)) a method which combines the Walsh-Hadamard transformation (WHT) with randomizing of the original data (files, images) by xoring with randomly chosen bits from random data that have been stored during a preprocessing stage. As it was shown, this model can be interpreted as a holographic one. It is 'holographic' because any portion of holographic coded information (which is any subset of corresponding codewords)represents a blurred image of the entire data. We consider the Hamming distance between original and reconstructed binary files as a 'blurreness' measure.

In this presentation we suggest a performance model for reconstruction a binary sequence from a truncated WH series. In order to estimate the performance of the method we consider the theoretic Shannon bound $R = -Dlog_2(D) - (1-0.D)log_2(1-D)$, where D is the fraction (probability) of the correctly reconstructed bits and R is the number of bits per symbol transmitted. For example, if the symbol is one bit, and $D = 1/2$, then $R = 1$, implying that each transmitted bit has to be known (considered) explicitly.

The Walsh-Hadamard transformation is based on a complete set of orthogonal functions. That is, if $b = (b_1, b_2, \ldots, b_m)$ is a binary file (a binary sequence, or a binary vector), then n-character encoding of the file b can be represented as $c^T = W^T b$, where $c = (c_1, \ldots, c_n)$, $c = 2^k$, k is an integer, are the Walsh-Hadamard coefficients. These orthogonal functions use only the values 1 or -1. More detailed, the spectral coefficients of WHT are $c_h = (1/n) \sum_{i=0}^{n-1} b_i W(h,i)$, and the inverse transform is $b_i = \sum_{h=0}^{n-1} c_i W(h,i)$.

Let $b = b_0, b_2 \ldots, b_{n-1}$ be an uncorrelated ("'white-noise'-like") sequence of n bits (generating by xoring mentioned above), where n is a power of two integer, and, due to the uncorrelation, $Prob(b_i = 1)$=$Prob(b_i = 0)$=1/2.

Let us we use for the original sequence reconstruction only $l \ll n$ WHT coefficients $c_1, \ldots, c_l$. In this case, we can estimate each bit b_i of the randomized sequence b by WHT mentioned above as $\hat{b}_i = \tilde{b}_i + e_i(l)$, where $\tilde{b}_i = \sum_{j=0}^{l} c_j W(j,i)$, and $e_i(l) = \sum_{q=l+1}^{n-1} c_q W(q,i)$.

Our goal is to compute a metric that captures the difference of the bits b_i and $\tilde{b}_i$. The result may depend on the coefficients we choose for reconstruction, in dependency on the application requirements. Each coefficient c_i is transmitted/stored with its index i in the WHT matrix, namely the pairs $(c_i;\ i)$ are stored as the representation of the data. We may consider various ways of the l choice, for example, either random choice of l coefficients (which can be reasonable, say, for distributed communication channels), or using first greatest l coefficients. We grouned why the latter way is more reasonabile. Inverse WHT with partial sums may result in non-binary values, that differ from binary domain of original sequence. Therefore, the reconstruction metric should be considered along with a decision rule mapping each value to a corresponding binary value. We suggest to round the values to the closest value in the field during the decoding process.

The reconstructed estimation of a bit $b_i = round(\hat{b}_i)$, where $\hat{b}_i$ is the estimation of the i-th value before rounding, computed by a partial sum of inverse WHT, is determined by the following random events:

$e_0 : (b_i = 0), e_1 : (b_i = 1)$, that is the bit b_i of randomized file F is 0 (event e_0) or 1 (event e_1), $v_{i0} : \tilde{b}_i \leq 1/2, v_{i1} : \tilde{b}_i \geq 1/2$, (defined on the space of the rational values $\tilde{b}_i$).

Let $Pr_{err=0}(i)$ be the probability that the actually zero bit b_i was erroneously reconstructed as $b_i = 1$, and $Pr_{err=1}(i)$ be the probability that the bit $b_i = 1$ was erroneously reconstructed as $b_i = 0$.

Both the probabilities $Prob(v_{i0})$, $Prob(v_{i1})$ are the probabilities of the partial sums mentioned above that have a value that can be estimated to be close to 1/2. Formally, in order to estimate error of the sequence reconstruction by truncated number of coefficients we should know both joint and marginal distributions both the sum of l terms of the WHT $S_l = \sum_{j=0}^{l} c_j W(j,i))$ and sum of residue $S_R = \sum_{j=n-l+1}^{N} c_j W(j,i)$. Then, taking into account that the sum $S_l + S_R$ is an exact value $b_i = 0$ or 1, we could characterize the error probability by the $Prob(S_l \geq Tr/S_l + S_R = 0)$, $Prob(S_l \leq Tr/S_l + S_R = 1)$. In accordance with Theorem 6.4 in P. A. Morettin (1981), WHT coefficients are distributed (asymptotically) as some independent normal random values with zero mean and dispersion of $n \times f(i)$, where i is the WHT coefficient index and $f(i)$ is the (dyadic) spectral density of b. Note, that there is an ambiguity in the definition of choice of l largest coefficients if there exist pair of coefficients c_i, c_j, such that $abs(c_i) = abs(c_j)$. It is possible to use an identification of all WHT coefficients indexes that contribute significantly to the binary sequences energy, that is the sum of the sequence of Boolean ones.

Acknowledgment. The second author has partially been supported by the Russian Foundation for Basic Research under grant RFBR No. 12-07-00109.

References

1.

2. S. Dolev, S. Frenkel. A way of coding and decoding of digital data based on digital holograpphy principles. *Patent of Russian Federation 2010145892/08(066164)* of 11.11.2010.

3. P. A. Morettin. Walsh Spectral Analysis. *SIAM Review*, vol. 23, pp. 277-291, 1981.

Analysis of M/G/1 queue with hysteretic load control

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Overload control is critical in preventing congestion in modern switching networks. A simple and intuitively appealing technique to detect congestion is a queue-length threshold. Such a mechanism is hysteretic control which is proposed to use by IETF (Internet Engeneering Task Force) to prevent overload in SIP (Session Initiation Protocol) signalling networks [1-4].

We consider a variant of hysteretic load control mechanism with three thresholds – congestion onset threshold H, congestion abatement threshold L and load discard threshold R (fig. 1). The mechanism functions as follows: when the buffer occupancy exceeds threshold H, congestion is detected and load is reduced to avoid overloading. To avoid oscillations between functioning modes load is not recovered immediately after buffer occupancy is decreased to H, but only when it falls to threshold L. Similarly, if buffer occupancy in congested mode reaches threshold R the load is discarded and recovers to congested mode value only when it falls below H.

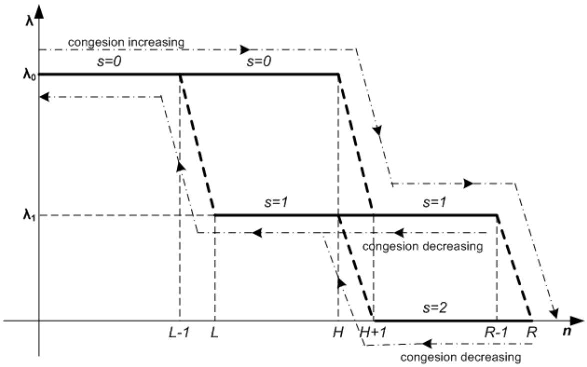

Figure 1: Hysteretic load control.

To obtain more general results we describe the system behavior in terms of $M_2|G|1|\langle L,H\rangle|\langle H,R\rangle$ queue. Similar model with just two thresholds and infinite queue size ($R=\infty$) is analysed by Roughan and Pearce [5] using a martingale technique. For our model we provide a system of equations for steady-state probability distribution using Markov renewal processes technique as described in [6]. In addition, we derive formulas and get numerical results for several system characterestics that are of interest considering hysteretic load control mechanism:

- the probability that the system is in congestion mode;
- the probability that the system is in discard mode;
- the average control cycle time;
- the average time spent in congestion and discard modes.

Acknowledgements. This work was supported in part by the Russian Foundation for Basic Research (grants 10-07-00487 and 12-07-00108).

References

1. J. Rosenberg. Requirements for Management of Overload in the Session Initiation Protocol. *RFC 5390*, December 2008.

2. V. Hilt, E. Noel, C. Shen, A. Abdelal. Design Considerations for Session Initiation Protocol (SIP) Overload Control. *RFC 6357*, August 2011.

3. S. Montagna, M. Pignolo. Load control techniques in SIP signalling servers using multiple thresholds. *13th International Telecommunications Network Strategy and Planning Symposium, NETWORKS*, 2008, pp. 1–17.

4. P. O. Abaev, Y. V. Gaidamaka, K. E. Samouylov. Hysteretic Overload Control in a SIP Signalling Network. *Vestnik RUDN, Series Mathematics, Informatics, Physics*, 2011, No. 4, pp. 55–73.

5. M. Roughan, C. E. M. Pearce. A Martingale Analysis of Hysteretic Overload Control. *Advances in Performance Analysis*, 2000, Vol. 3, No. 1, pp. 1–30.

6. E. Sopin. Analysis of $M^{[x]}/G/1/r$ queue with a resume level. *XXIX International Seminar on Stability Problems for Stochastic Models. Book of Abstracts*, 2011, pp. 53–55.

Some conditions of adaptive strategies existence

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Adaptive strategy is considered as a controlling algorithm which achieves the goal while interacting with arbitrary object from the given class. There are many constructions of such algorithms as well as many mathematical results that state their optimality. Similar results are nothing else than sufficient conditions of adaptive strategy existence. This report is aimed to make some remarks about necessary conditions of adaptive controllability in the indicated area.

The mathematical model exploits the discrete time “subject-object interaction“ scheme. Here the “object“ is given as the sequence of controllable conditional distributions which define state transitions and the ”subject” is associated with the strategy – that is a sequence of conditional probabilities, which sets the rules of control choice at each point of time. As the strategy is adapted to the object with a priori unknown characteristics so it should “learn“ itself in-control basing on the observable part of object prehistory.

Naturally no “irreparable destroying faults“ may be done in process of training. The requirement of “learning without destruction“ ability is a base for the first type of necessary existence conditions. Another idea is not so obvious and consists in the presence of not more as countable set of “variants“ from which the optimal one can be selected for arbitrary object from the given class.

The report contains several examples illustrating necessary conditions and the theorem presenting the criteria of adaptive strategy existence for special class of random controllable sequences. The proof of the theorem uses so called “adaptive enumeration strategy“ [1]. The full text of the report is about to be published in the scientific journal “Systems and Means of Informatics“.

References

1. M. G. Konovalov. *Methods of adaptive information processing and their application*, 2007, Institute of Informatics Problems RAS, ISBN 978-5-902030-59-9 (in Russian).

Stationary waiting time distribution in Markov queueing system with ordinary and negative customers, bunker and different service rates under LAST-LIFO-LIFO discipline

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Consideration is given to single-line queueing system with Poisson incoming flow of customers. Henceforth these customers are called ordinary. For ordinary customers, there is buffer with infinite capacity. Besides ordinary customers, Poisson stream of negative customers enters the system. A negative customer entering the system takes one ordinary customer from the end of the queue in the buffer and transfers in into another queue (bunker) which also has infinite capacity (negative customer itself leaves the system after making a transfer). If there are no ordinary customers in the buffer at the moment when a negative customer arrives at the system, then it leaves the system and does not influence it in any way. The system chooses customers for servicing as follows. After a customer has been serviced, server chooses for service the last customer in the queue in the buffer. If the buffer is empty at the moment of service completion, the customer, which is last standing in the queue in the buffer, goes to server. The servicing process is not interrupted by either ordinary or negative customers. Service times of customers from buffer and bunker have exponential distribution but with different service rates.

This research continues the work begun in [1], which is devoted to analysis of the same queueing system but with equal service rates for ordinary and negative customers. Introduction of unequal service rates results in serious complication of the situation which, in turn, leads to the fact that the method used in [1] cannot be applied. Therefore, new method was proposed that allows one to find in terms of Laplace-Stieltjes transform stationary waiting time distribution of an arriving ordinary customer and distribution of the busy period of the considered system. Noteworthy, that LST of the busy period is expressed as a functional equation which is impossible to invert as well as in case of $M|G|1$ queueing system.

This work was partially supported by Russian Foundation for Basic Research, project No 11-07-00112 and No 12-07-00108.

References

1. A. V. Pechinkin, R. V. Razumchik. Exponential Queueing System with Negative Customers and Bunker for Ousted Customers. *Automation and Remote Control*, 2011, vol. 72, No. 12, pp. 2492–2504.

Software tools for spherical stochastic systems analysis and filtering

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Software tools for nonlinear Euclidian and multichannel circular stochastic systems (CStS) analysis and filtering are described in [1-4]. The paper is devoted to the corresponding software tools for spherical StS (SphStS) based on equvalent statistical linearization.

Let us consider SphStS described by the following nonlinear Ito stochastic differential Eqs:

$$\dot{\Theta}_1 = \varphi_1^{\Theta}(\Theta_1, \Theta_2, X, t), \qquad \dot{\Theta}_2 = \varphi_2^{\Theta}(\Theta_1, \Theta_2, X, t), \tag{1}$$

$$\dot{X} = \varphi^X(\Theta_1, \Theta_2, X, t) + \psi^X(\Theta_1, \Theta_2, X, t)V. \tag{2}$$

Here Θ_1, Θ_2 being spherical scalar variables (SphV); X being linear vector instrumental variable (LV); $\varphi_i^{\Theta}(\Theta_1, \Theta_2, X, t)(i = 1, 2)$ being scalar nonlinear functions; $\varphi^X(\Theta_1, \Theta_2, X, t)$ being vector nonlinear function; $\psi^X(\Theta_1, \Theta_2, X, t)$ being matrix nonlinear function; V being vector white noise (the derivative of vector process with independent increments) with known matrix intensity

$\nu(t)$. Using the (by mean square criteria) equivalent statistical linearization (ESL) of nonlinear functions in Eqs (1), (2):

$$\varphi(\Theta_1, \Theta_2, X, t) = \varphi_0 + k^{\Theta_1}\Theta_1^0 + k^{\Theta_2}\Theta_2^0 + k^X X^0, \tag{3}$$

where $\varphi_0, k^{\Theta_1}, k^{\Theta_2}, k^X$ being coefficients of equivalent statistical linearization depending on parametres equivalent probability LV and SphV density, we get deterministic Eqs for mathematical expectations and quasilinear Eqs for centred SphV Θ_i^0 and LV X^0. Then on the basis of linear StS theory [1,5] we get variances and covariances deterministic Eqs for times t and t' for off-line data analysis. Using [5] for Eqs (1), (2) together with Eqs for observed SphV we get corresponding Eqs for on-line quazilinear filtering.

Analogously discrete SphStS are considered. "Wrapped" normal densities for statistical linearization in state and filtering Eqs [1,5] are implemented.

The original software tools "SphStS-filter" is instrumented in MATLAB for nonlinear discrete and continuous SphStS. Its current experimental version uses functions of MATLAB Symbolic Math toolbox and presents the set of open program functions with numerical and graphic output.

Applications: statistical dynamics of inertial sensors based on spherical pendulum and gyros [1,6-8].

The work is supported by Russian Foundation for Basic Research (Project #10-07-00021).

References

1. V. S. Pugachev, I. N. Sinitsyn. *Stochastic Systems. Theory and Applications*, 2001, World Scientific, Singapore.

2. I. N. Sinitsyn. Stochastic informational technologies for nonlinear circular stochastic systems research. *Informatics and applications*, 2011, vol. 5, No. 4, p. 78–89 (In Russian).

3. I. N. Sinitsyn. Mathematical software for nonlinear multichannel circular stochastic systems analysis based on distribution parametrization. *Informatics and applications*, 2012, vol. 6, No. 1, p. 11–17 (In Russian).

4. I. N. Sinitsyn, E. R. Korepanov, V. V. Belousov, T. D. Konashenkova. The development of mathematical software for multichannel nonlinear circular stochastic systems analysis. *Systems and means of Informatics*, 2012, vol. 22, No. 1, p. 3–21 (In Russian).

5. I. N. Sinitsyn. *Kalman and Pugachev filters*, 2007, LOGOS, Moscow (2nd edition, in Russian).

6. I. N. Sinitsyn, V. V. Belousov, T. D. Konashenkova. Software tools for Circular Stochastic Systems Analysis. // In Book of Abstracts "XXIX International Seminar on Stability Problems for Stochastic Models" and V International Workshop "Applied Problems in Theory of Probabilities and Mathematical Statistics related to modeling of information systems", 10–16 october 2011, Svetlogorsk, Russia, p. 86–87.

7. N. I. Fisher, T. Lewis, B. J. J. Embleton. *Statistical analysis of spherical data*, 1993, Cambridge University Press.

8. V. S. Pugachev (Ed.). *Automation Control Foundation.*, 1974, Nauka, Moscow (In Russian).

The heavy traffic limiting distribution of the waiting time in a priority queue with hyperexponential input stream

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The sequence of the single server queues with hyperexponential input stream and head-of-the-line priority discipline is considered. Entering customers in n-th system are separated into r priority classes with probability $p_1^{(n)}, \ldots, p_r^{(n)}$. Customers of i-th class have priority to customers of j-th class if $i < j$. Service times are jointly independent random variables with distribution function $B_i^{(n)}(x)$ for customers of $i-$th class,

$$\beta_i^{(n)}(s) = \int_0^\infty e^{-sx} dB_i^{(n)}(x), \ \beta_{ij}^{(n)} = \int_0^\infty x^j dB_i^{(n)}(x).$$

Let $a^{(n)}(x) = \sum\limits_{j=1}^{N} c_j^{(n)} a_j^{(n)} \exp\left(-a_j^{(n)} x\right), x \geqslant 0, \ a_i^{(n)} \neq a_j^{(n)}, \ i \neq j, \ c_j^{(n)} > 0, \ \sum\limits_{i=1}^{N} c_i^{(n)} = 1$, be the density of interarrival time, $w^{(n)}(t)$ – the virtual waiting time for lowest priority class at time t in n-th system,

$$a^{(n)} = \left(\sum_{j=1}^{N} c_j^{(n)} \left(a_j^{(n)}\right)^{-1}\right)^{-1}, \ \rho_{k1}^{(n)} = a^{(n)} \cdot \sum_{i=1}^{k} p_i^{(n)} \beta_{i1}^{(n)}, \ \rho_{k2}^{(n)} = a^{(n)} \cdot \sum_{i=1}^{k} p_i^{(n)} \beta_{i2}^{(n)},$$

$$\rho_k^{(n)} = 1 - \rho_{k1}^{(n)}, \ \rho^{(n)} = \rho_r^{(n)}, \ u^{(n)} = \frac{\rho_{r2}^{(n)}}{2} + a^{(n)} \cdot \sum_{j=1}^{N} c_j^{(n)} \left(a_j^{(n)}\right)^{-2} - \sum_{j=1}^{N} c_j^{(n)} \left(a_j^{(n)}\right)^{-1}.$$

Assume that:

1)
$$\beta_i^{(n)}(s) = 1 - \beta_{i1}^{(n)}\, s + \frac{1}{2}\,\beta_{i2}^{(n)}\, s^2 + o_n(s^2),$$
where $\dfrac{o_n(s^2)}{s^2} \to 0$ as $s \to 0$;

2) for all $n \geqslant 1$ $\rho_{r1}^{(n)} < 1$;

3) $\lim\limits_{n\to\infty} c_j^{(n)} = c_j$, $\lim\limits_{n\to\infty} a_j^{(n)} = a_j$, $j = 1, \ldots, N$, $\lim\limits_{n\to\infty} \beta_{ik}^{(n)} = \beta_{ik}$, $k = 1, 2$, $i = 1, \ldots, r$, $\lim\limits_{n\to\infty} p_i^{(n)} = p_i$, $i = 1, \ldots, r$, $\lim\limits_{n\to\infty} \rho_{r-11}^{(n)} < 1$, $\lim\limits_{n\to\infty} \rho_{r1}^{(n)} = 1$, $\lim\limits_{n\to\infty} u^{(n)} = u$.

Theorem.
$$\lim_{n\to\infty} \mathbf{P}\left(\left(\rho^{(n)}\right)^{\delta} w^{(n)}\left(t\left(\rho^{(n)}\right)^{-\alpha}\right) < x\right) =$$
$$= \begin{cases} \sqrt{2\pi} \displaystyle\int_0^{\sqrt{2ut^{-1}}\ k} e^{-\frac{y^2}{2}}\, dy, \ \alpha < 2, \\ 1 - \pi^{-\frac{1}{2}} \left(e^{-2k} \displaystyle\int_{-\sqrt{\frac{t}{4u}} + k\cdot\sqrt{\frac{u}{t}}}^{+\infty} e^{-y^2} dy + \displaystyle\int_{\sqrt{\frac{t}{4u}} + k\cdot\sqrt{\frac{u}{t}}}^{+\infty} e^{-y^2} dy \right), \ \alpha = 2, \\ 1 - e^{-2k}, \ \alpha > 2, \end{cases}$$
where
$$k = \frac{\rho_{r-1}}{2u}\, x, \quad \delta = \begin{cases} \dfrac{\alpha}{2}, & \alpha \leqslant 2, \\ 1, & \alpha > 2. \end{cases}$$

References

1. V. F. Matveev, V. G. Ushakov. *Queueing systems*, 1984, Moscow university, Moscow.

Null ergodicity bounds for a class of queueing models

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Markov chain $X = X(t)$, $t \geq 0$ is called *null ergodic*, if $Pr\,\{X(t) = i\} \to 0$ as $t \to \infty$ for any initial conditions and any i.

Null ergodicity and related bounds for birth-death queueing models have been studied from 1990-s, see for instance [1-4].

Here we consider null ergodicity for more general class of nonstationary queueing systems with batch arrivals and group services.

Let $X = X(t)$, $t \geq 0$, be a number of customers in the queueing system ($0 \leqslant X(t) < \infty$).

Denote by $p_{ij}(s,t) = Pr\left\{X(t) = j\left|X(s) = i\right.\right\}$, $i,j \geqslant 0$, $0 \leq s \leq t$ and by $p_i(t) = Pr\left\{X(t) = i\right\}$ the transition and state probabilities of $X = X(t)$ respectively.

We suppose that

$$\Pr\left(X\left(t+h\right) = j/X\left(t\right) = i\right) = \begin{cases} q_{ij}\left(t\right)h + \alpha_{ij}\left(t,h\right) & \text{if } j \neq i \\ 1 - \sum\limits_{k \neq i} q_{ik}\left(t\right)h + \alpha_i\left(t,h\right) & \text{if } j = i, \end{cases}$$

where all $\alpha_i(t,h)$ are $o(h)$ uniformly in i, i. e. $\sup_i |\alpha_i(t,h)| = o(h)$.

We also suppose $q_{i,i+k}\left(t\right) = \lambda_k(t)$, $q_{i+k,i}\left(t\right) = \mu_k(t)$ for any $k > 0$.

In other words, we will suppose that arrival rates $\lambda_k(t)$ and service rates $\mu_k(t)$ do not depend on the the length of queue. In addition, we assume that $\lambda_{k+1}(t) \leqslant \lambda_k(t)$ and $\mu_{k+1}(t) \leqslant \mu_k(t)$ for any k and almost all $t \geqslant 0$.

Then under standard assumptions (see [4]) the probabilistic dynamics of the process is represented by the forward Kolmogorov differential system:

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}(t),$$

where

$$A(t) = \begin{pmatrix} a_{00}(t) & \mu_1(t) & \mu_2(t) & \mu_3(t) & \mu_4(t) & \cdots & \mu_r(t) & \cdots \\ \lambda_1(t) & a_{11}(t) & \mu_1(t) & \mu_2(t) & \mu_3(t) & \cdots & \mu_{r-1}(t) & \cdots \\ \lambda_2(t) & \lambda_1(t) & a_{22}(t) & \mu_1(t) & \mu_2(t) & \cdots & \mu_{r-2}(t) & \cdots \\ \cdots \\ \lambda_r(t) & \lambda_{r-1}(t) & \lambda_{r-2}(t) & \cdots & \lambda_2(t) & \lambda_1(t) & a_{rr}(t) & \cdots \\ \cdots \end{pmatrix},$$

where $a_{ii}(t) = -\sum_{k=1}^{i} \mu_k(t) - \sum_{k=1}^{\infty} \lambda_k(t)$ and $\sup_i |a_{ii}(t)| < \infty$ for almost all $t \geqslant 0$.

We denote throughout the paper by $\| \bullet \|$ the l_1-norm, i. e. $\|\mathbf{x}\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^{\infty}$.

Let Ω be a set all stochastic vectors, i. e. l_1 vectors with nonnegative coordinates and unit norm.

Then we have $\|A(t)\| \leqslant 2\sum_{k=1}^{\infty}(\lambda_k(t) + \mu_k(t))$ for almost all $t \geqslant 0$. Hence operator function $A(t)$ from l_1 into itself is bounded for almost all $t \geqslant 0$ and locally integrable on $[0;\infty)$. Therefore we can consider (1) as a differential equation in the space l_1 with bounded operator.

Consider a sequence of positive numbers $\{d_i\}$, $i = 1, 2, \dots$ and put

$$\nu(t) = \inf_{i \geqslant 0} \left(|a_{ii}(t)| - \sum_{k=1}^{i} \frac{d_{i-k}}{d_i} \mu_k(t) - \sum_{k=1}^{\infty} \frac{d_{i+k}}{d_i} \lambda_k(t) \right).$$

Theorem 1. Let us assume that there exists a sequence of positive numbers $\{d_j\}$ such that $d_{-1} = d_0 = 1$, $\sup_{i \geqslant 1} d_i = \mathrm{d} < \infty$, and

$$\int_0^\infty \nu(t)\, dt = +\infty.$$

Then $X(t)$ is null ergodic, and the following bound holds:

$$\sum_{i=0}^{\infty} d_i p_i(t) \leqslant \mathrm{d} e^{-\int_s^t \nu(\tau)\, d\tau},$$

for any $0 \leqslant s \leqslant t$ and any n.

We also consider a class of such queueing systems and study their null ergodicity.

Acknowledgements This work was supported by the Russian Foundation for Basic Research, projects no. 11-01-12026, 12-07-00115, 12-07-00109.

References

1. B. L. Granovsky, A. I. Zeifman. Nonstationary Queues: Estimation of the Rate of Convergence. *Queueing Systems*, 2004, vol. 46, p. 363–388.

2. A. I. Zeifman. Some estimates of the rate of convergence for birth and death processes. *J. Appl. Probab.*, 1991, vol. 28, p. 268–277.

3. A. I. Zeifman. Upper and lower bounds on the rate of convergence for nonhomogeneous birth and death processes. *Stochastic Processes and their Applications*, 1995, vol. 59, p. 157–173.

4. A. I. Zeifman, V. E. Bening, I. A. Sokolov. *Continuous-time Markov chains and models (in Russian)*, 2008, Elex-KM, Moscow.

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